Spin-orbit coupled atomic Fermi gases

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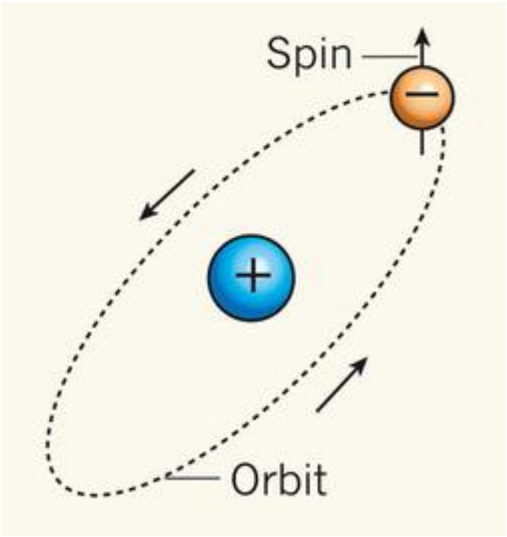
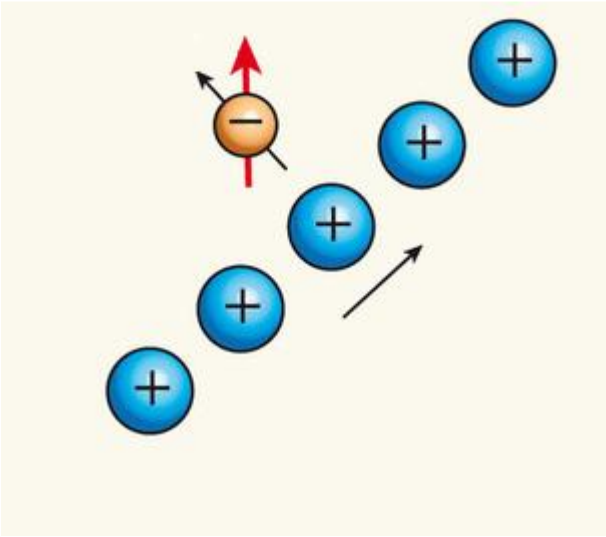
A$: "Foundation Project" @ACQAO, and ARC DP (QEII) Grants

Motivation: Spin-orbit coupling (SOC)

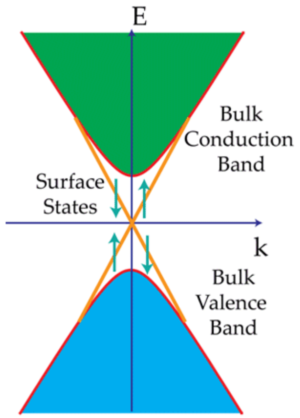
Atomic physics



Solid-state physics

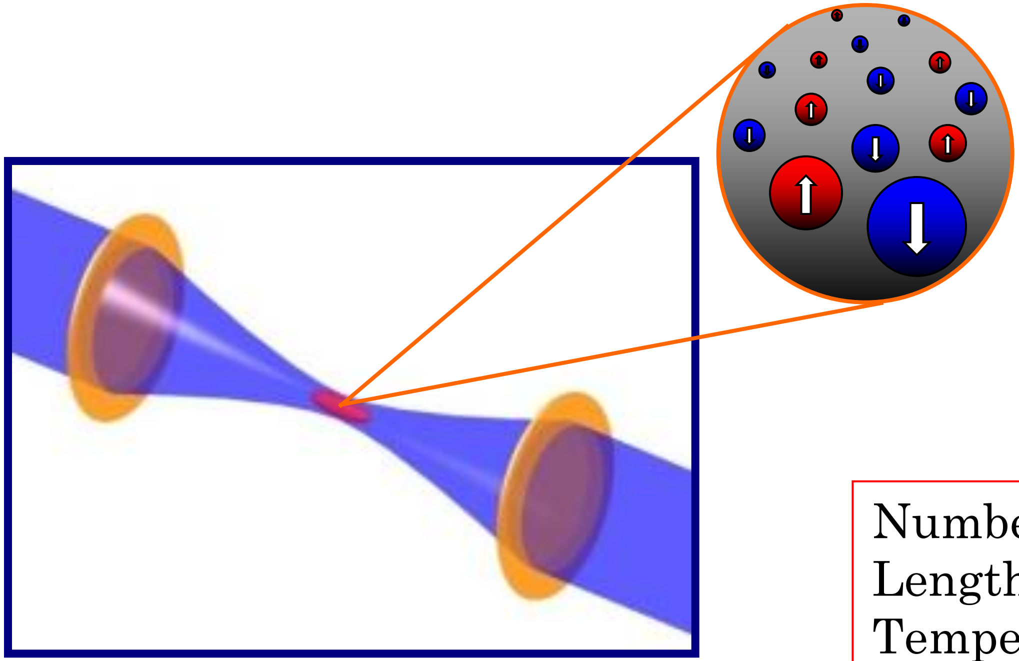


topological insulators



Spin-orbit coupling plays a key role in different branches of physics.

Motivation: SOC in neutral ultracold atoms?



Number of atoms: **10^4-10^6**
Length scale: **100 μm**
Temperature scale: **100 nK**
Interaction: ***s*-wave dominant**

Confined: **harmonic traps**

Ultracold atoms is an ideal table-top system for **exploring new states of matter**.

Toolkit: **Feshbach resonance + Optical lattice + Cavity + Disorder + SOC**

Motivation: SOC in neutral ultracold atoms?

LETTER

doi:10.1038/nature09887

Spin–orbit-coupled Bose–Einstein condensates

Y.-J. Lin[1], K. Jiménez-García[1,2] & I. B. Spielman[1]

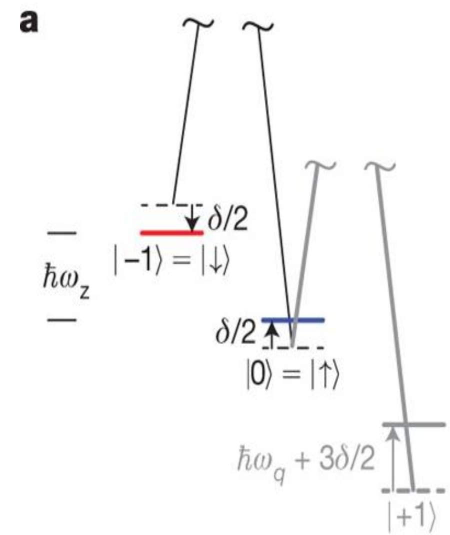
Spin–orbit (SO) coupling—the interaction between a quantum particle's spin and its momentum—is ubiquitous in physical systems. In condensed matter systems, SO coupling is crucial for the spin-Hall effect[1,2] and topological insulators[3–5]; it contributes to the electronic properties of materials such as GaAs, and is important for spintronic devices[6]. Quantum many-body systems of ultracold atoms can be precisely controlled experimentally, and would therefore seem to provide an ideal platform on which to study SO coupling. Although an atom's intrinsic SO coupling affects its electronic structure, it does not lead to coupling between the spin and the centre-of-mass motion of the atom. Here, we engineer SO coupling (with equal Rashba[7] and Dresselhaus[8] strengths) in a neutral atomic Bose–Einstein condensate by dressing two atomic spin states with a pair of lasers[9]. Such coupling has not been realized previously for ultracold atomic gases, or indeed any bosonic system. Furthermore, in the presence of the laser coupling, the interactions between the two dressed atomic spin states are modified, driving a quantum phase transition from a spatially spin-mixed state (lasers off) to a phase-separated state (above a critical laser intensity). We develop a many-body theory that provides quantitative agreement with the observed location of the transition. The engineered SO coupling—equally applicable for bosons and fermions—sets the stage for the realization of topological insulators in fermionic neutral atom systems.

α parametrizes the SO-coupling strength; $\Omega = -g\mu_B B_z$ and $\delta = -g\mu_B B_y$ result from the Zeeman fields along $\hat{z}$ and $\hat{y}$, respectively; and $\check{\sigma}_{x,y,z}$ are the 2×2 Pauli matrices. Without SO coupling, electrons have group velocity $v_x = \hbar k_x/m$, independent of their spin. With SO coupling, their velocity becomes spin-dependent, $v_x = \hbar(k_x \pm 2\alpha m/\hbar^2)/m$ for spin $|\uparrow\rangle$ and $|\downarrow\rangle$ electrons (quantized along $\hat{y}$). In two recent experiments, this form of SO coupling was engineered in GaAs heterostructures where confinement into two-dimensional planes linearized the native cubic SO coupling of GaAs to produce a Dresselhaus term, and asymmetries in the confining potential gave rise to Rashba coupling. In one experiment a persistent spin helix was found[6], and in another the SO coupling was only revealed by adding a Zeeman field[10].

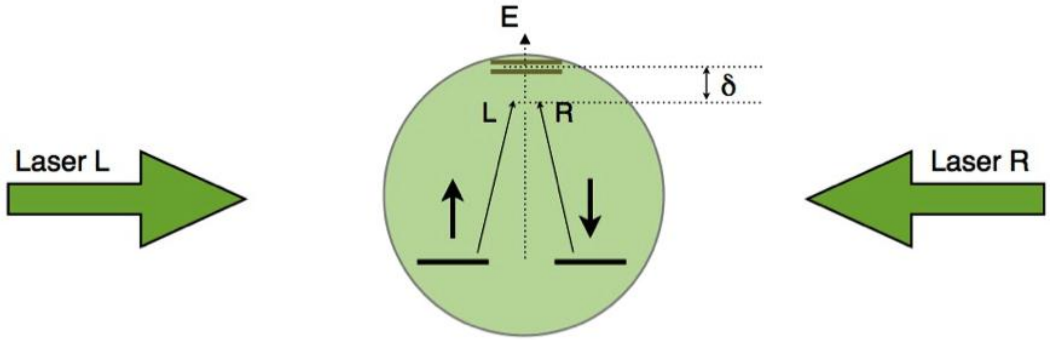
SO coupling for neutral atoms enables a range of exciting experiments, and importantly, it is essential in the realization of neutral atom topological insulators. Topological insulators are novel fermionic band insulators including integer quantum Hall states and now spin quantum Hall states that insulate in the bulk, but conduct in topologically protected quantized edge channels. The first-known topological insulators—integer quantum Hall states[11]—require large magnetic fields that explicitly break time-reversal symmetry. In a seminal paper[3], Kane and Mele showed that in some cases SO coupling leads to zero-magnetic-field topological insulators that preserve time-reversal symmetry. In the absence of the bulk conductance that plagues current materials, cold atoms can potentially realize such an insulator

Y.-J. Lin *et al.*, *Nature* **471**, 83 (2011) (3 March 2011)

Raman process



Motivation: SOC in neutral ultracold atoms?



Physics — *Physics* **5**, 96 (2012)

Viewpoint

Spin-Orbit Coupling Comes in From the Cold

Erich J. Mueller
Laboratory of Atomic and Solid State Physics, Cornell University, Ithaca, NY 14853, USA

Published August 27, 2012

Experimentalists simulate the effects of spin-orbit coupling in ultracold Fermi gases, paving the way for the creation of new exotic phases of matter.

Subject Areas: **Atomic and Molecular Physics**

A Viewpoint on:
Spin-Orbit Coupled Degenerate Fermi Gases
Pengjun Wang, Zeng-Qiang Yu, Zhengkun Fu, Jiao Miao, Lianghui Huang, Shijie Chai, Hui Zhai, and Jing Zhang
Phys. Rev. Lett. **109**, 095301 (2012) – Published August 27, 2012

Spin-Injection Spectroscopy of a Spin-Orbit Coupled Fermi Gas
Lawrence W. Cheuk, Ariel T. Sommer, Zoran Hadzibabic, Tarik Yefsah, Waseem S. Bakr, and Martin W. Zwierlein
Phys. Rev. Lett. **109**, 095302 (2012) – Published August 27, 2012

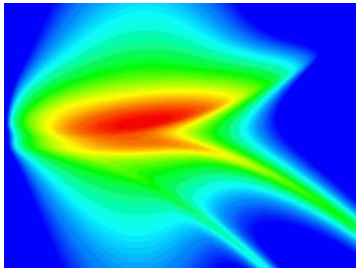
Ian Spielman group: PRL (2013).

Outline of the lectures: exotic superfluids

- Experimental realization of SOC and two-body study (I & II)

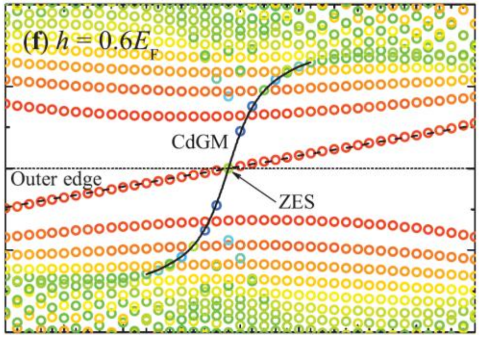
(No Zeeman field)

- Anisotropic superfluidity



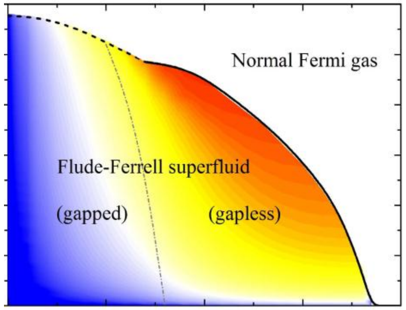
(Out-of-plane Zeeman field)

- Topological superfluid and Majorana fermions



(In-plane Zeeman field)

- Fulde-Ferrell superfluidity



Reference: review chapter

CHAPTER 2

FERMI GASES WITH SYNTHETIC SPIN–ORBIT COUPLING

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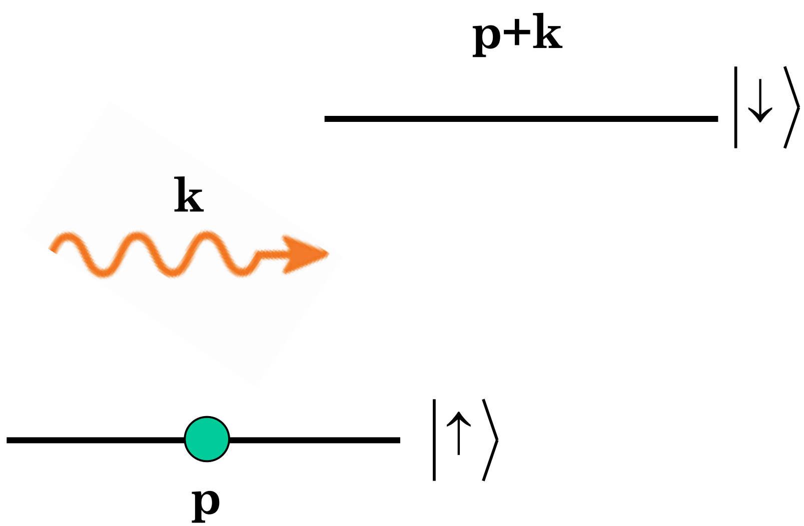
Annual Review of Cold Atoms and Molecules, Vol. 2, 2014

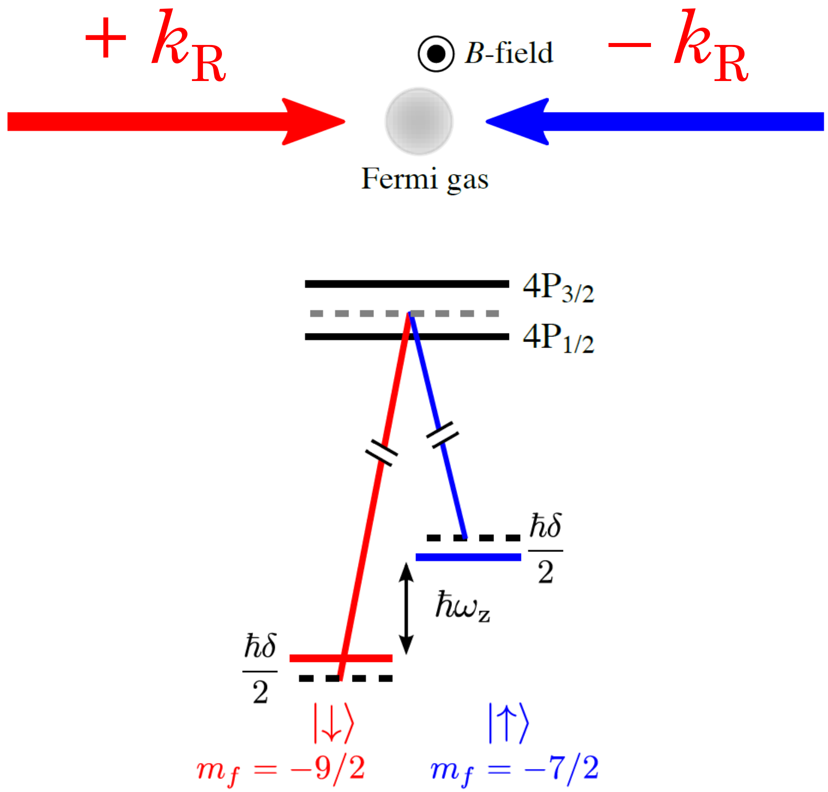
Lecture I: few-body physics

Simple idea of spin-orbit coupling



Here, unlike electrons, we don't care about the real spin of atoms. When we say "spin", we refer to the **hyperfine states** that atoms stay.

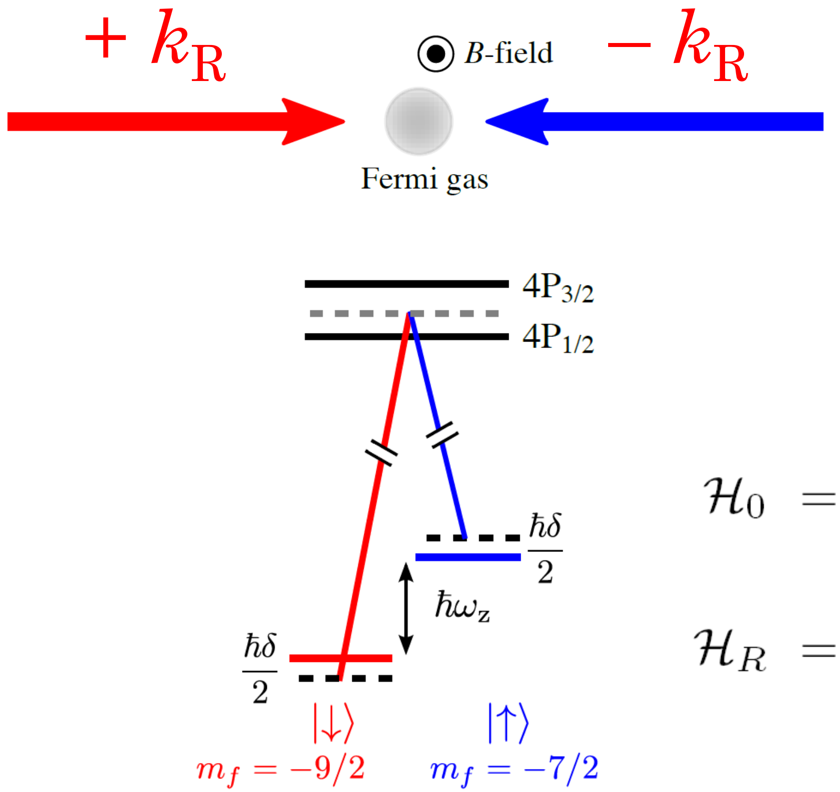
Realization of SOC in neutral ultracold atoms



Ian Spielman group: PRL (2013).

$k_R = 2\pi/\lambda$ is determined by the wave length λ of two lasers and $2\hbar k_R$ is the momentum transfer during the two-photon Raman process

Realization of SOC in neutral ultracold atoms



Ian Spielman group: PRL (2013).

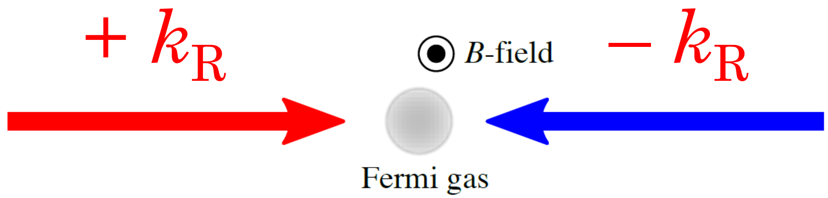
$$\mathcal{H}_0 = \sum_\sigma \int d\mathbf{r}\psi_\sigma^\dagger(\mathbf{r})\frac{\hbar^2k^2}{2M}\psi_\sigma(\mathbf{r}),$$

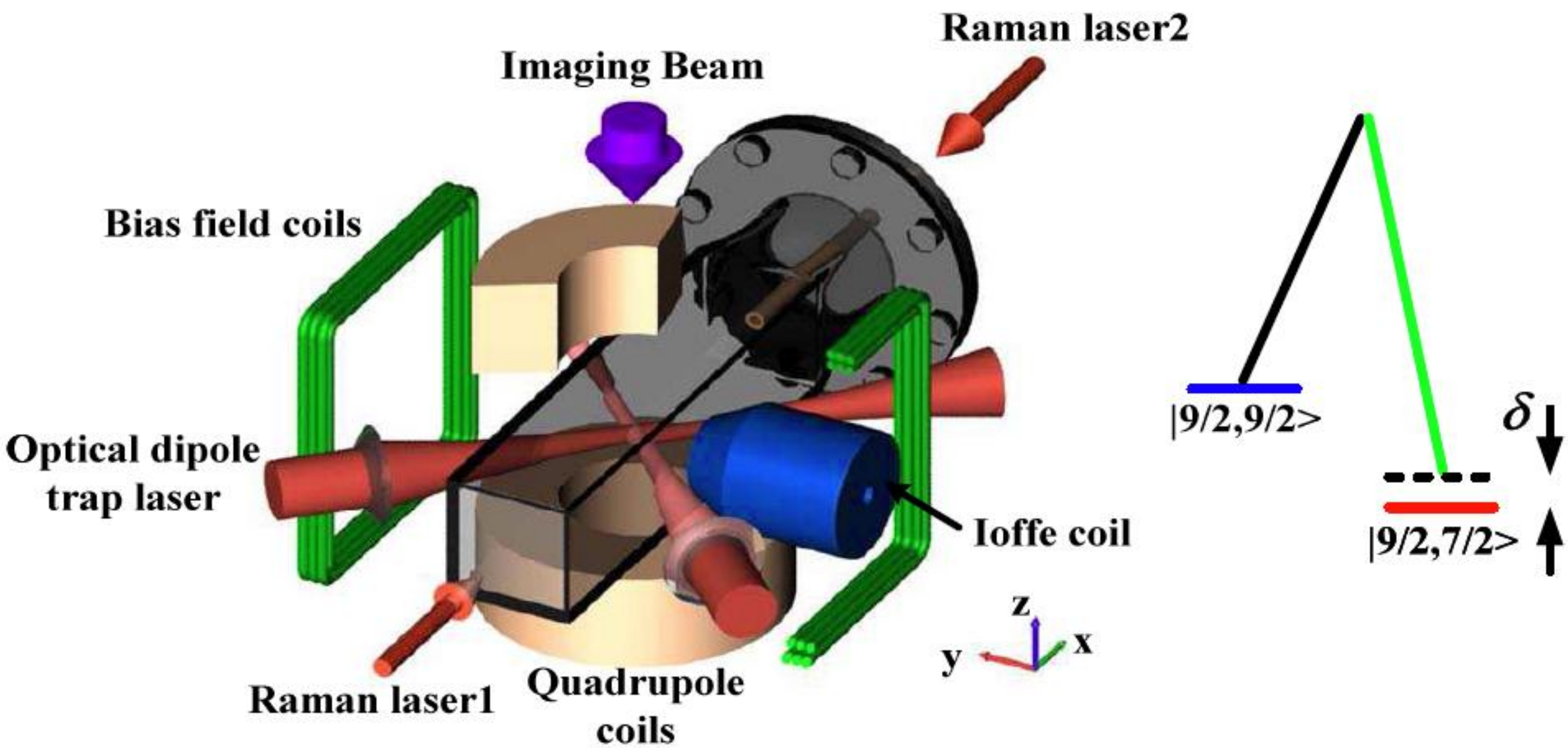
$$\mathcal{H}_R = \frac{\Omega_R}{2}\int d\mathbf{r}\left[\psi_\uparrow^\dagger(\mathbf{r})e^{i2k_Rx}\psi_\downarrow(\mathbf{r})+\text{H.c.}\right],$$

(SOC at δ=0)

$k_R = 2\pi/\lambda$ is determined by the wave length λ of two lasers and $2\hbar k_R$ is the momentum transfer during the two-photon Raman process

Realization of SOC in neutral ultracold atoms





P. Wang *et al.*, *PRL* **109**, 095301 (2012). **Shanxi University, China**

Realization of SOC in neutral ultracold atoms

$$\mathcal{H}_0 = \sum_\sigma \int d\mathbf{r} \psi_\sigma^\dagger(\mathbf{r}) \frac{\hbar^2 k^2}{2M} \psi_\sigma(\mathbf{r}),$$

$$\mathcal{H}_R = \frac{\Omega_R}{2} \int d\mathbf{r} \left[\psi_\uparrow^\dagger(\mathbf{r}) e^{i2k_R x} \psi_\downarrow(\mathbf{r}) + \text{H.c.} \right]$$

(gauge transformation):

$$\psi_\uparrow(\mathbf{r}) = e^{+ik_R x} \tilde{\psi}_\uparrow(\mathbf{r}),$$

$$\psi_\downarrow(\mathbf{r}) = e^{-ik_R x} \tilde{\psi}_\downarrow(\mathbf{r}),$$

$$\mathcal{H}_0 = \sum_\sigma \int d\mathbf{r} \left[\tilde{\psi}_\sigma^\dagger(\mathbf{r}) \frac{\hbar^2 (\mathbf{k} \pm k_R \mathbf{e}_x)^2}{2M} \tilde{\psi}_\sigma(\mathbf{r}) \right]$$

$$\mathcal{H}_R = \frac{\Omega_R}{2} \int d\mathbf{r} \left[\tilde{\psi}_\uparrow^\dagger(\mathbf{r}) \tilde{\psi}_\downarrow(\mathbf{r}) + \text{H.c.} \right]$$

Realization of SOC in neutral ultracold atoms

$$\Phi(\mathbf{r}) \equiv [\tilde{\psi}_{\uparrow}(\mathbf{r}), \tilde{\psi}_{\downarrow}(\mathbf{r})]^T$$

$$\boxed{\mathcal{H} = \int d\mathbf{r}\Phi^{\dagger}(\mathbf{r})\left[H_{SO} \quad\right]\Phi(\mathbf{r}),}$$

$$H_{SO} \equiv \frac{\hbar^2\left(k_R^2 + \mathbf{k}^2\right)}{2M} + h\sigma_x + \lambda k_x \sigma_z \qquad H_0 + H_R$$

Here, for convenience we have introduced a spin-orbit coupling constant $\lambda \equiv \hbar^2 k_R / M$, an "effective" Zeeman field $h \equiv \Omega_R / 2$, and an "effective" lattice depth $V_L \equiv \Omega_{RF} / 2$.

Realization of SOC in neutral ultracold atoms

$$H_{SO} \equiv \frac{\hbar^2\left(k_R^2 + \mathbf{k}^2\right)}{2M} + h\sigma_x + \lambda k_x \sigma_z$$

(gauge transformation):

$$\tilde{\psi}_{\uparrow}(\mathbf{r}) = \frac{1}{\sqrt{2}}\left[\Psi_{\uparrow}(\mathbf{r}) - i\Psi_{\downarrow}(\mathbf{r})\right],$$

$$\tilde{\psi}_{\downarrow}(\mathbf{r}) = \frac{1}{\sqrt{2}}\left[\Psi_{\uparrow}(\mathbf{r}) + i\Psi_{\downarrow}(\mathbf{r})\right],$$

Equal Rashba and Dresselhaus SOC !!!

$$\boxed{V_{\mathrm{SO}} = h\sigma_z + \lambda k_x \sigma_y = \frac{\Omega_{\mathrm{R}}}{2}\sigma_z + \frac{\hbar^2 k_R}{M} k_x \sigma_y}$$

Recall that in solid state:

$$V_{\mathrm{SO}} = \lambda_{\mathrm{R}}\left(+k_y\sigma_x - k_x\sigma_y\right)$$ **Rashba spin-orbit coupling**

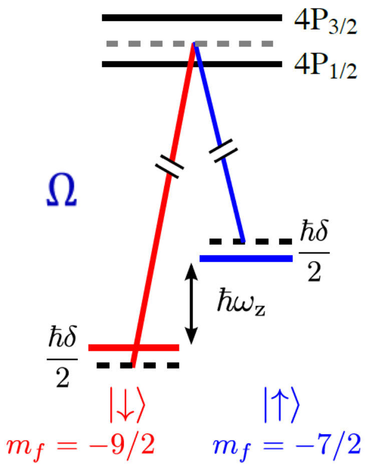
$$V_{\mathrm{SO}} = \lambda_{\mathrm{D}}\left(-k_y\sigma_x - k_x\sigma_y\right)$$ **Dresselhaus spin-orbit coupling**

Importantly:

$$\mathcal{H}_{int} = U_0 \int d\mathbf{r} \psi_{\uparrow}^{\dagger}(\mathbf{r}) \psi_{\downarrow}^{\dagger}(\mathbf{r}) \psi_{\downarrow}(\mathbf{r}) \psi_{\uparrow}(\mathbf{r})$$

- **The form of the interaction Hamiltonian is not changed by two gauge transformation;**
- **The terms without spin-flip remains the same;**
- **The momenta of the basis for spin-up and spin–down atoms are shifted by $\pm k_{\mathrm{R}}$.**
- **$\Omega_R = 0$ means no spin-orbit coupling!**

Realization of SOC in neutral ultracold atoms



$$\lambda_{SO} = \frac{\hbar^2 k_R}{2m}$$

out-of-plane Zeeman field

$$\boxed{H = \frac{\hbar^2 \hat{k}^2}{2m} + \underline{\lambda_{SO} k_x \sigma_y} + \frac{\delta}{2}\sigma_y + \frac{\Omega}{2}\sigma_z}$$

in-plane Zeeman field

One-dimensional spin-orbit coupling so far! But already rich physics.

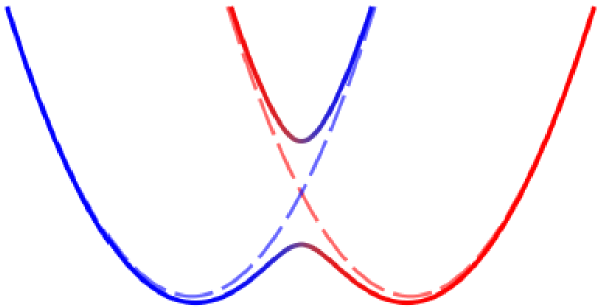
Single-particle state

$$\boxed{H_{SO} \equiv \frac{\hbar^2\left(k_R^2 + \mathbf{k}^2\right)}{2M} + h\sigma_x + \lambda k_x \sigma_z}$$

SOC at δ=0,
forget the trapping potential …

The model Hamiltonian H_{SO} describes a spin-orbit coupling with equal Rashba and Dresselhaus strengths [2, 5–7]. The single-particle solution $\phi_{\mathbf{k}}(\mathbf{r})$ satisfies the Schrödinger equation, $H_{SO}\phi_{\mathbf{k}}(\mathbf{r}) = \epsilon_{\mathbf{k}}\phi_{\mathbf{k}}(\mathbf{r})$. Using the Pauli matrices and the fact that the wave-vector or momentum $\mathbf{k} \equiv (k_x, \mathbf{k}_\perp) \equiv (k_x, k_y, k_z)$ is a good quantum number, it is easy to see that we have two eigenvalues

$$\epsilon_{\mathbf{k}\pm} = \frac{\hbar^2 k_\perp^2}{2M} + \frac{\hbar^2\left(k_R^2 + k_x^2\right)}{2M} \pm \sqrt{h^2 + \lambda^2 k_x^2},$$



where "±" stands for two helicity branches. The corresponding eigenstates are given by (we set the volume $V = 1$),

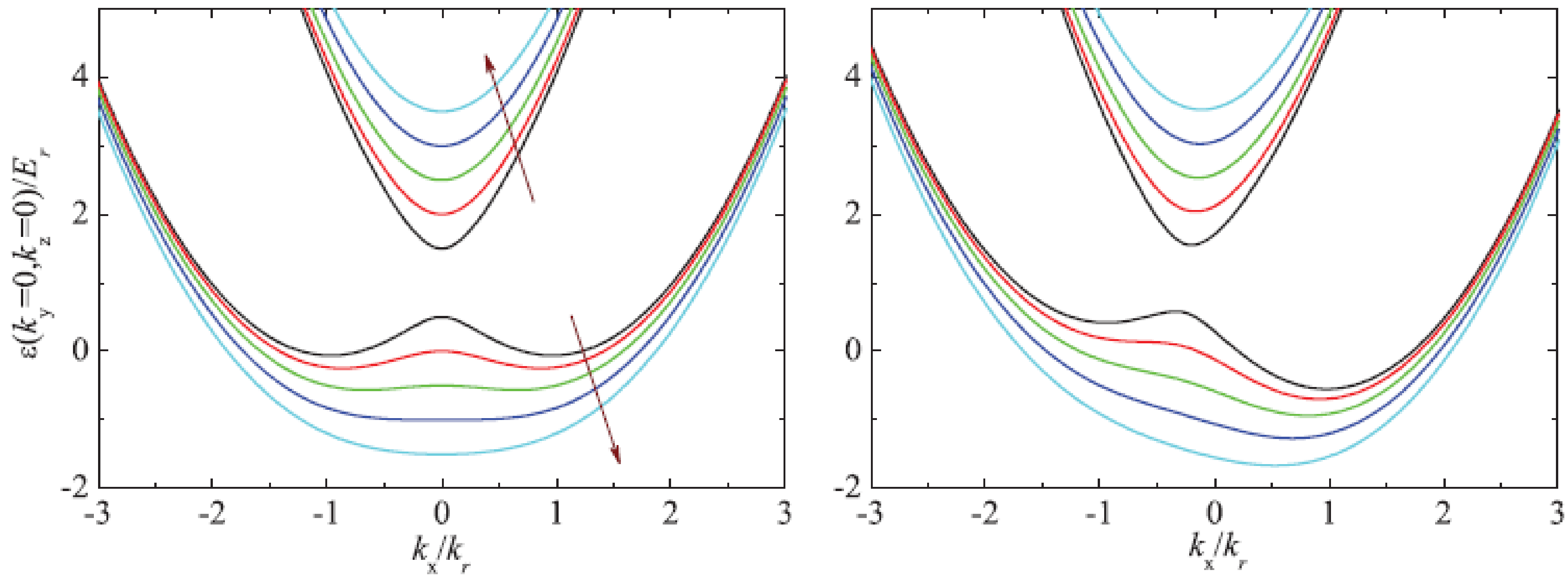
$$\phi_{\mathbf{k}}^{(+)}(\mathbf{r}) = \left[\begin{pmatrix} \cos\theta_{\mathbf{k}} \\ \sin\theta_{\mathbf{k}} \end{pmatrix} e^{ik_x x}\right] e^{i\mathbf{k}_\perp \cdot \mathbf{r}_\perp},$$

$$\phi_{\mathbf{k}}^{(-)}(\mathbf{r}) = \left[\begin{pmatrix} -\sin\theta_{\mathbf{k}} \\ \cos\theta_{\mathbf{k}} \end{pmatrix} e^{ik_x x}\right] e^{i\mathbf{k}_\perp \cdot \mathbf{r}_\perp},$$

where $\underline{\theta_{\mathbf{k}} = \arctan[(\sqrt{h^2 + \lambda^2 k_x^2} - \lambda k_x)/h]}$ and $\mathbf{r}_\perp \equiv (y, z)$.

Single-particle state

$$\ldots \pm \sqrt{h^2 + (\lambda k_x)^2} \qquad \ldots \pm \sqrt{h^2 + (\lambda k_x + \delta/2)^2}$$



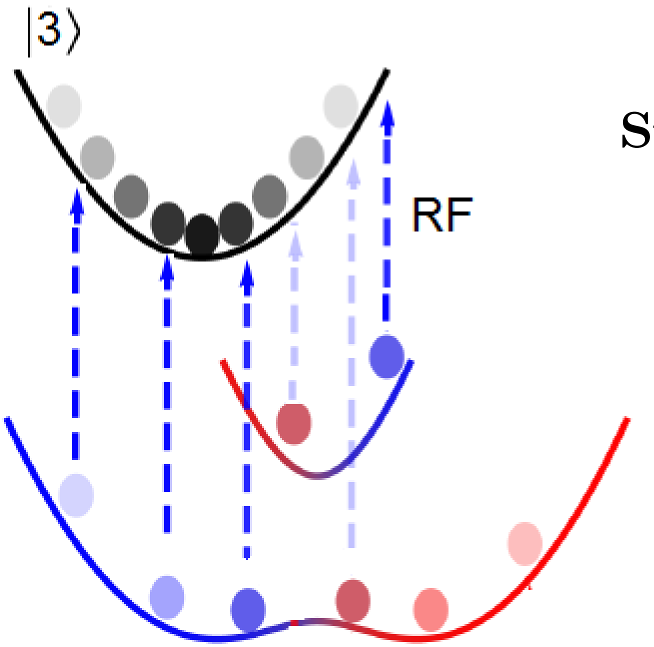
(a) $\delta = 0$ (b) $\delta = E_r$

$$E_R = \frac{(\hbar k_R)^2}{2m}$$

Single-particle state (rf-spectroscopy)

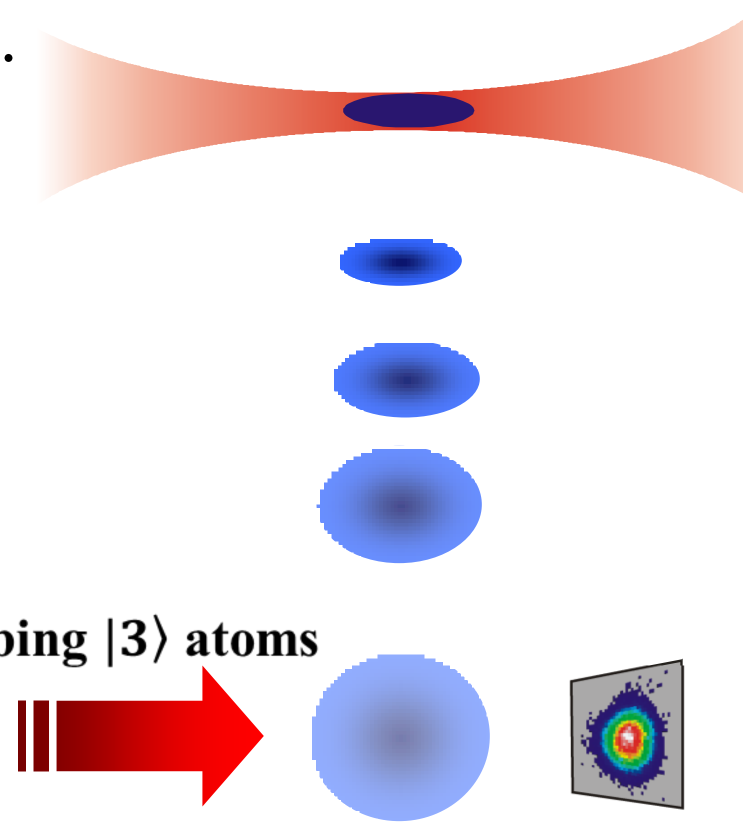
momentum-resolved radio-frequency (rf) spectroscopy

Step I.



Step II.

Time-of-flight absorption imaging

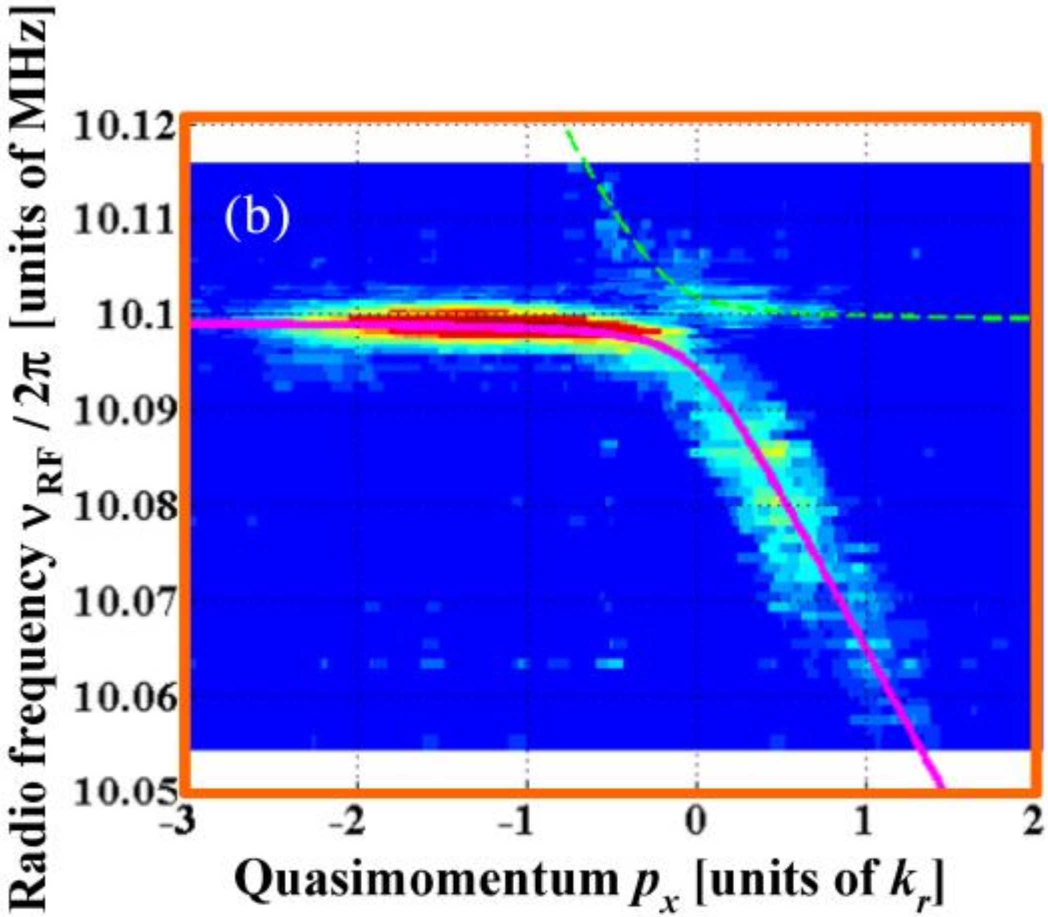


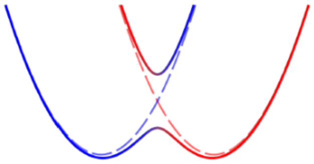
Probing |3⟩ atoms

$$\mathcal{V}_{rf} = V_0 \int d\mathbf{r} \left[\psi_3^{\dagger}(\mathbf{r}) \psi_{\downarrow}(\mathbf{r}) + \psi_{\downarrow}^{\dagger}(\mathbf{r}) \psi_3(\mathbf{r}) \right]$$
$$= V_0 \int d\mathbf{r} \left[e^{-ik_R x} \psi_3^{\dagger}(\mathbf{r}) \tilde{\psi}_{\downarrow}(\mathbf{r}) + \mathrm{H.c.} \right]$$

Ideally, measure the single-particle spectral function $A(\mathbf{k},\omega)$

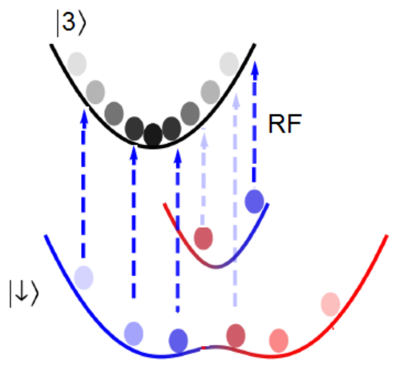
Single-particle state (rf-spectroscopy)





Observation at Shanxi University!

Single-particle state (rf-spectroscopy)

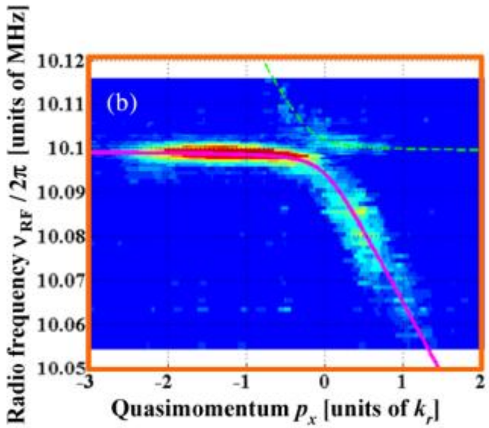


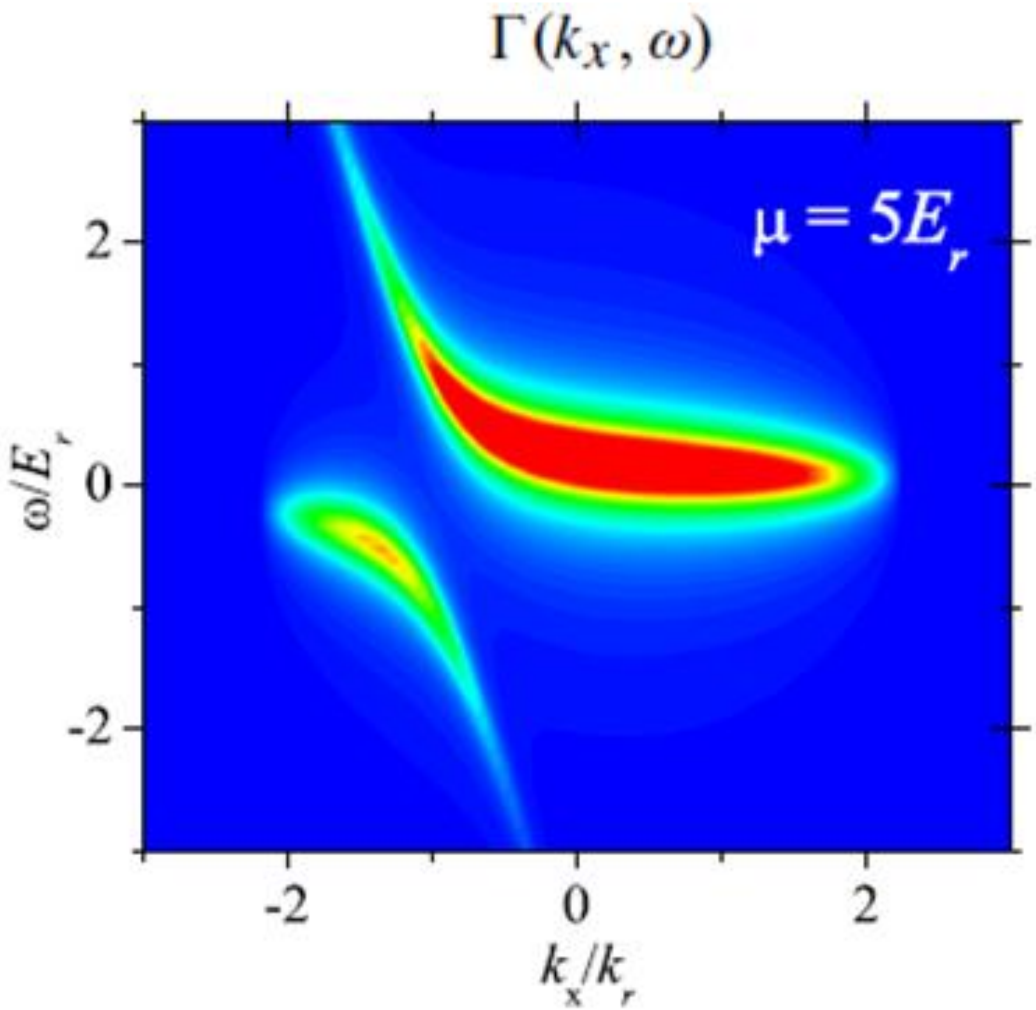
The Fermi golden rule for rf-transfer strength:

$$\Gamma(\omega) = \sum_{i,f} \left| \langle \Phi_f | \mathcal{V}_{rf} | \Phi_i \rangle \right|^2 f(E_i - \mu)\, \delta\left[\hbar\omega - \cancel{\hbar\omega_{3\downarrow}} - (E_f - E_i)\right]$$

Here, the summation is over all the possible initial states **i** (with energy E_i) and final states **f** (with energy E_f) and $f\,(E_i - \mu)$ is the Fermi distribution function. The Dirac delta function ensures energy conservation during transition.

Single-particle state (rf-spectroscopy)



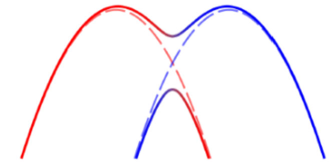


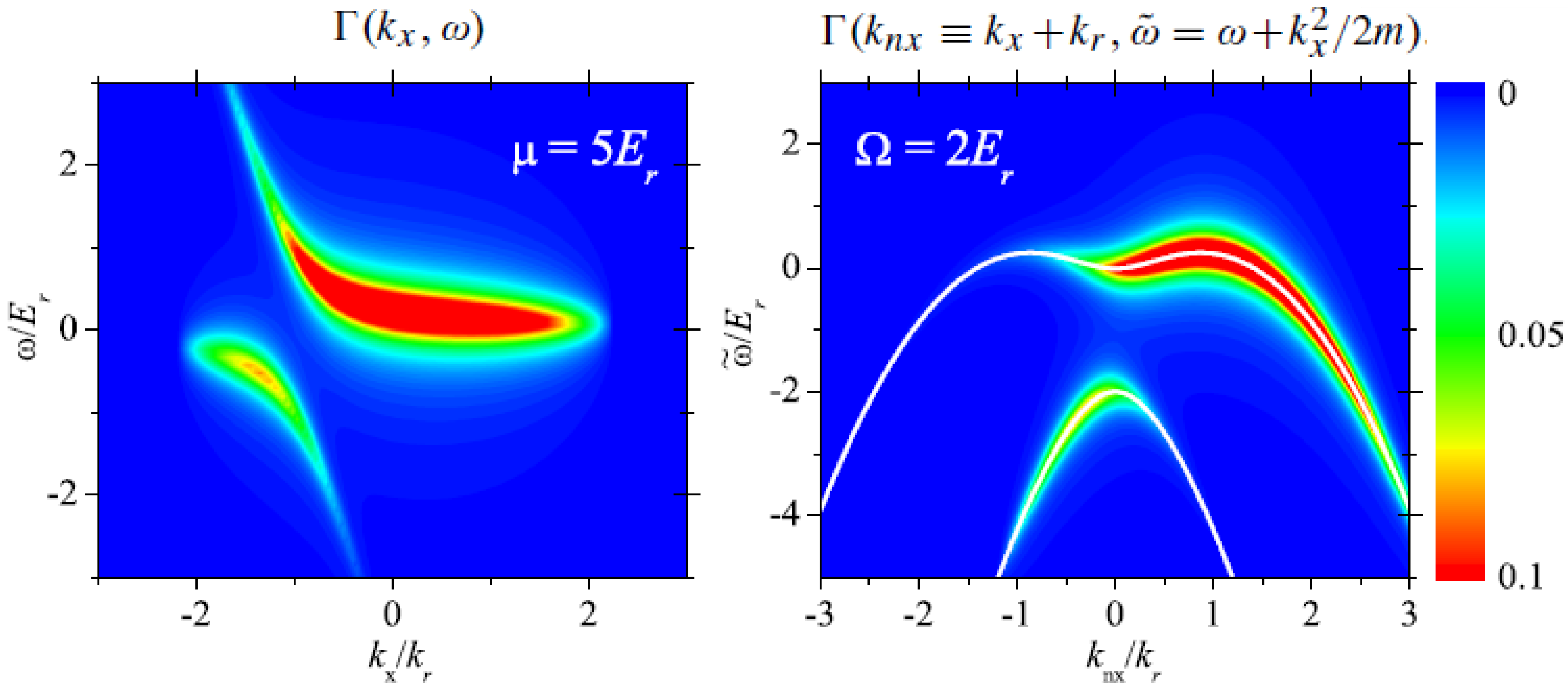
Single-particle state (rf-spectroscopy)

Key factors to understand the spectrum:

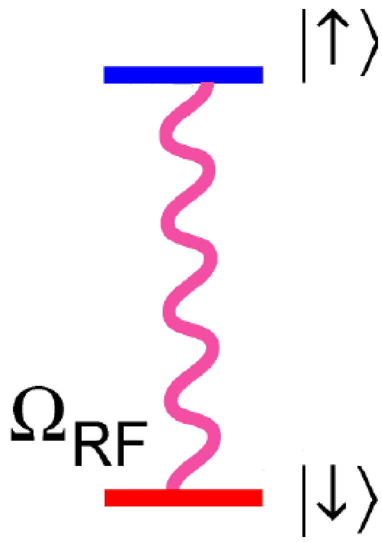
- The momentum of the basis for spin–down atoms is shifted by $-\boldsymbol{k}_{\mathrm{R}}$;
- Energy conservation $\delta\left[\omega-\left(E_f-E_i\right)\right]$;
- The transfer strength is proportional to the amplitude of spin-down component;
- Don't worry about the trap; local density approximation works pretty good for N~10^5.

Observed at Shanxi University!





Theoretical simulation on momentum-resolved rf spectroscopy of a Fermi gas with 1D equal-weight Rashba–Dresselhaus SOC. Left panel: simulated experimental spectroscopy $\Gamma(k_x, \omega)$. Right panel: the spectroscopy $\Gamma(k_{nx} \equiv k_x + k_r, \tilde{\omega} = \omega + k_x^2/2m)$. Here, the intensity of the contour plot shows the number of transferred atoms, increasing linearly from 0 (blue) to its maximum value (red). We have set $\omega_{3\downarrow} = 0$ and used a Lorentzian distribution to replace the Delta function.



$$\mathcal{H}_{RF} = \frac{\Omega_{RF}}{2}\int d\mathbf{r}\left[\psi_{\uparrow}^{\dagger}(\mathbf{r})\,\psi_{\downarrow}(\mathbf{r}) + \mathrm{H.c.}\right]$$

(It is responsible for a **SOC** lattice!)

L. W. Cheuk *et al.*, *PRL* **109**, 095302 (2012). **MIT**

Realization of SOC in neutral ultracold atoms

$$
\begin{aligned}
\mathcal{H}_0 &= \sum_\sigma \int d\mathbf{r} \psi_\sigma^\dagger(\mathbf{r}) \frac{\hbar^2 k^2}{2M} \psi_\sigma(\mathbf{r}), \\
\mathcal{H}_R &= \frac{\Omega_R}{2} \int d\mathbf{r} \left[\psi_\uparrow^\dagger(\mathbf{r}) e^{i2k_R x} \psi_\downarrow(\mathbf{r}) + \mathrm{H.c.} \right] \\
\mathcal{H}_{RF} &= \frac{\Omega_{RF}}{2} \int d\mathbf{r} \left[\psi_\uparrow^\dagger(\mathbf{r}) \psi_\downarrow(\mathbf{r}) + \mathrm{H.c.} \right]
\end{aligned}
$$

(gauge transformation):

$$
\begin{aligned}
\psi_\uparrow(\mathbf{r}) &= e^{+ik_R x} \tilde{\psi}_\uparrow(\mathbf{r}), \\
\psi_\downarrow(\mathbf{r}) &= e^{-ik_R x} \tilde{\psi}_\downarrow(\mathbf{r}),
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{H}_0 &= \sum_\sigma \int d\mathbf{r} \left[\tilde{\psi}_\sigma^\dagger(\mathbf{r}) \frac{\hbar^2 (\mathbf{k} \pm k_R \mathbf{e}_x)^2}{2M} \tilde{\psi}_\sigma(\mathbf{r}) \right] \\
\mathcal{H}_R &= \frac{\Omega_R}{2} \int d\mathbf{r} \left[\tilde{\psi}_\uparrow^\dagger(\mathbf{r}) \tilde{\psi}_\downarrow(\mathbf{r}) + \mathrm{H.c.} \right] \\
\mathcal{H}_{RF} &= \frac{\Omega_{RF}}{2} \int d\mathbf{r} \left[\tilde{\psi}_\uparrow^\dagger(\mathbf{r}) e^{-i2k_R x} \tilde{\psi}_\downarrow(\mathbf{r}) + \mathrm{H.c.} \right]
\end{aligned}
$$

Realization of SOC in neutral ultracold atoms

$$\Phi(\mathbf{r}) \equiv [\tilde{\psi}_\uparrow(\mathbf{r}), \tilde{\psi}_\downarrow(\mathbf{r})]^T$$

$$\boxed{\mathcal{H} = \int d\mathbf{r}\Phi^\dagger(\mathbf{r})\left[H_{SO} + V_L(x)\right]\Phi(\mathbf{r}),}$$

$$H_{SO} \equiv \frac{\hbar^2\left(k_R^2 + \mathbf{k}^2\right)}{2M} + h\sigma_x + \lambda k_x \sigma_z$$

$$V_L(x) \equiv V_L\left[\cos(2k_R x)\sigma_x + \sin(2k_R x)\sigma_y\right].$$

Here, for convenience we have introduced a spin-orbit coupling constant $\lambda \equiv \hbar^2 k_R/M$, an "effective" Zeeman field $h \equiv \Omega_R/2$, and an "effective" lattice depth $V_L \equiv \Omega_{RF}/2$.

X.-J. Liu, *PRA* **86**, 033613 (2012).

Single-particle state

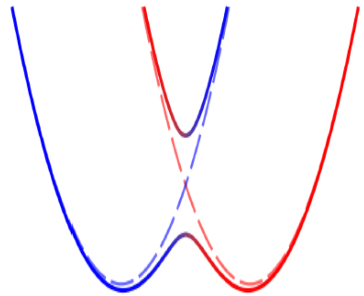
$$\boxed{V_L(x) \equiv V_L\left[\cos(2k_R x)\,\sigma_x + \sin(2k_R x)\,\sigma_y\right]}.$$

In the presence of the additional rf Hamiltonian $\mathcal{H}_{RF}$, the momentum along the x-axis, k_x, is no longer a good quantum number. The lattice potential terms $\cos(2k_R x)$ and $\sin(2k_R x)$ will couple the eigenstates $\phi^{(\pm)}_{\mathbf{k}'}(\mathbf{r})$ and $\phi^{(\pm)}_{\mathbf{k}''}(\mathbf{r})$ if $k'_x - k''_x = 2nk_R$, where $n = \pm1, \pm2, \cdots$ is an integer. In this case, it is useful to define a quasi-momentum or lattice momentum q_x for arbitrary k_x as follows: $k_x = 2nk_R + q_x$, where the integer n is chosen to make $-k_R \le q_x < k_R$. The quasi-momentum q_x is then a good quantum number. We may expand the single-particle eigenstate of the total Hamiltonian in the form,

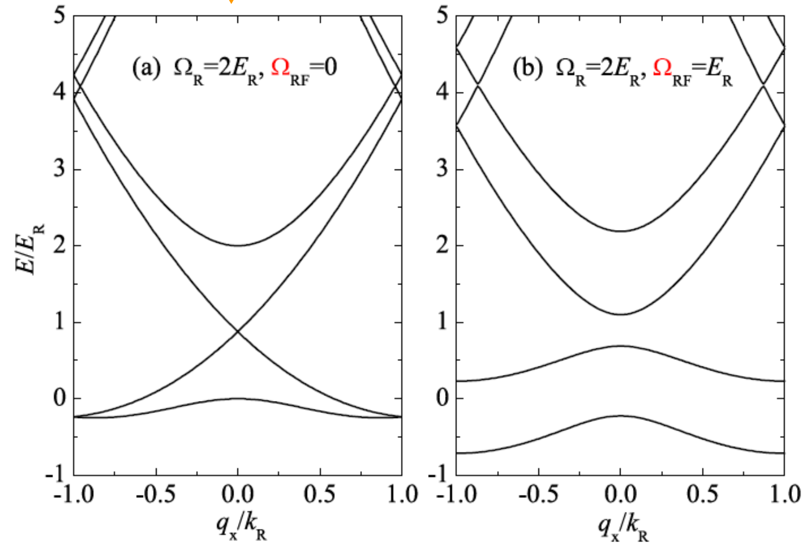
$$\Phi(q_x, \mathbf{k}_\perp; \mathbf{r}) = \sum_{m=-\infty}^{+\infty}\left[a_{m+}\phi^{(+)}_{\mathbf{k}_m}(\mathbf{r}) + a_{m-}\phi^{(-)}_{\mathbf{k}_m}(\mathbf{r})\right],$$

where $\mathbf{k}_m \equiv \mathbf{k}_\perp + (2mk_R + q_x)\mathbf{e}_x \equiv \mathbf{k}_\perp + k_{mx}\mathbf{e}_x$ has the same quasi-momentum q_x, and the energies of $\phi^{(+)}_{\mathbf{k}_m}(\mathbf{r})$ and $\phi^{(-)}_{\mathbf{k}_m}(\mathbf{r})$ are given by

$$\epsilon_{m\pm} \equiv \frac{\hbar^2 k_\perp^2}{2M} + \frac{\hbar^2\left(k_R^2 + k_{mx}^2\right)}{2M} \pm \sqrt{h^2 + \lambda^2 k_{mx}^2}.$$



folding into bands

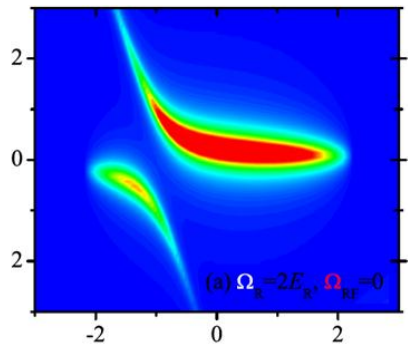


$$E^{(l)}(q_x)$$

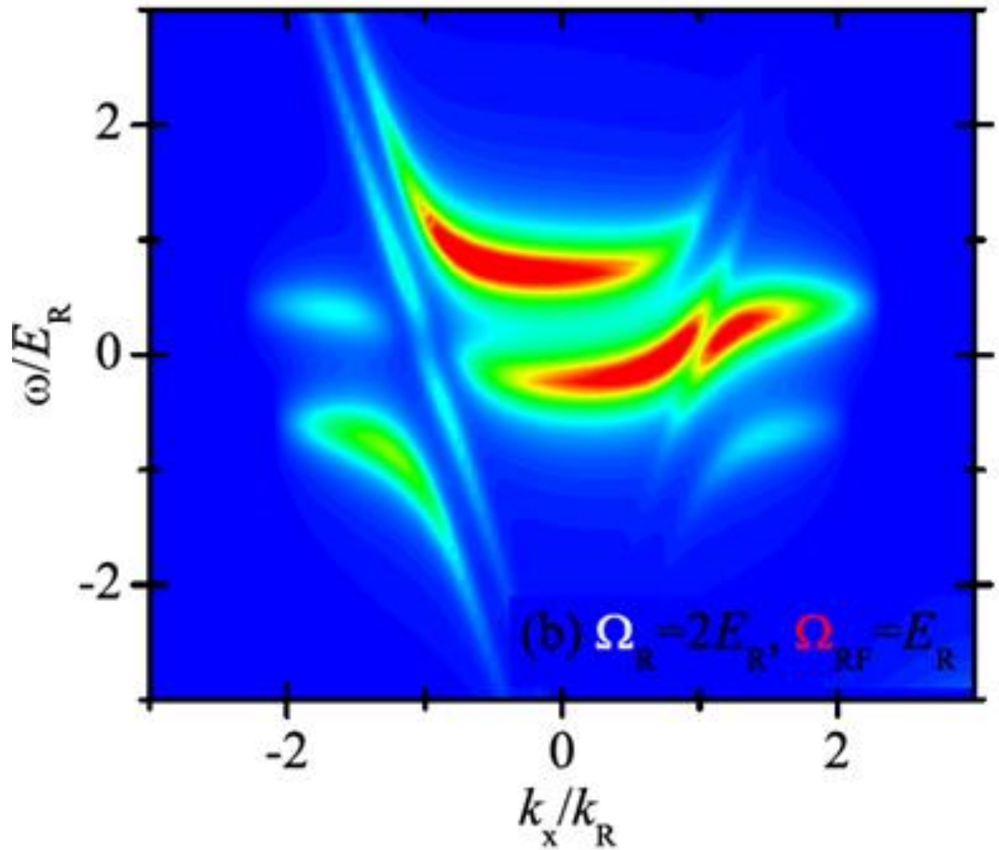
Single-particle state (rf-spectroscopy)

Momentum-resolved rf-transfer strength:

$$\Gamma\left(k_x,\omega\right)=\frac{Mk_BT}{4\pi^2\hbar^2}\sum_{l=0}^{\infty}\left[a_{n+}^{(l)}\sin\theta_{\mathbf{k}_n}+a_{n-}^{(l)}\cos\theta_{\mathbf{k}_n}\right]^2\ln\left\{1+\exp\left[-\frac{E^{(l)}(q_x)-\mu}{k_BT}\right]\right\}\delta\left[\hbar\omega+E^{(l)}(q_x)-\frac{\hbar^2k_x^2}{2M}\right]$$



without lattice

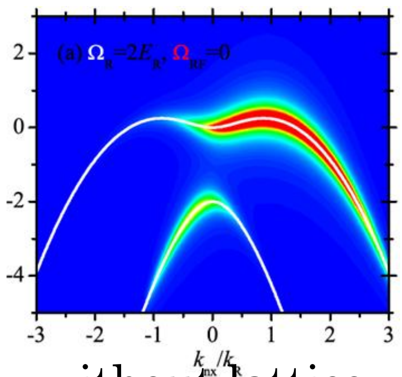


Observed at MIT using spin-injection spectroscopy

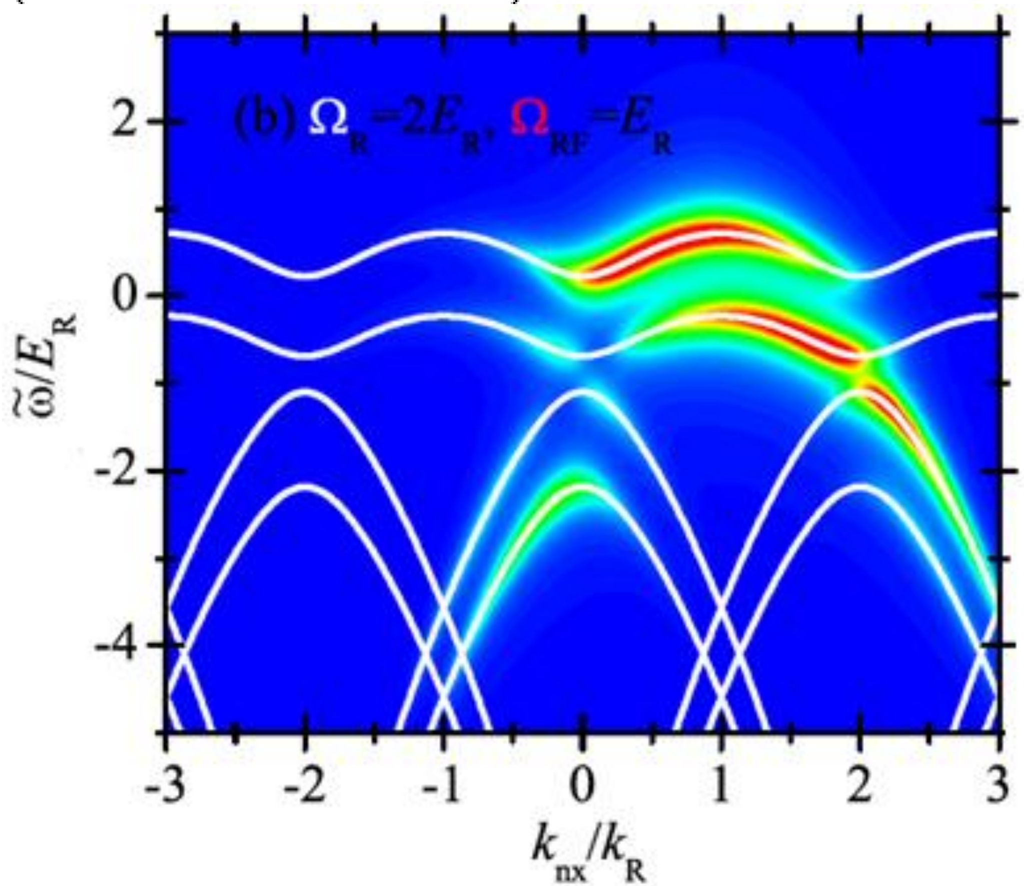
Single-particle state (rf-spectroscopy)

Momentum-resolved rf transfer strength:

$$\tilde{\Gamma}\left(k_{nx}, \tilde{\omega}\right) \equiv \Gamma\left(k_x + k_R, \omega + \frac{\hbar k_x^2}{2M}\right) \propto \delta[\hbar\tilde{\omega} + E^{(l)}(q_x)]$$



without lattice



X.-J. Liu, *PRA* **86**, 033613 (2012).

ARPES analogue (in solid state)

Atomic topological insulator?

Oleg Sushkov's lecture II:

2D **triangle** lattice + **spin-orbit coupling** = **topological insulator**

some **kinds of** lattices + **Raman coupling** = **atomic topological insulator** ?

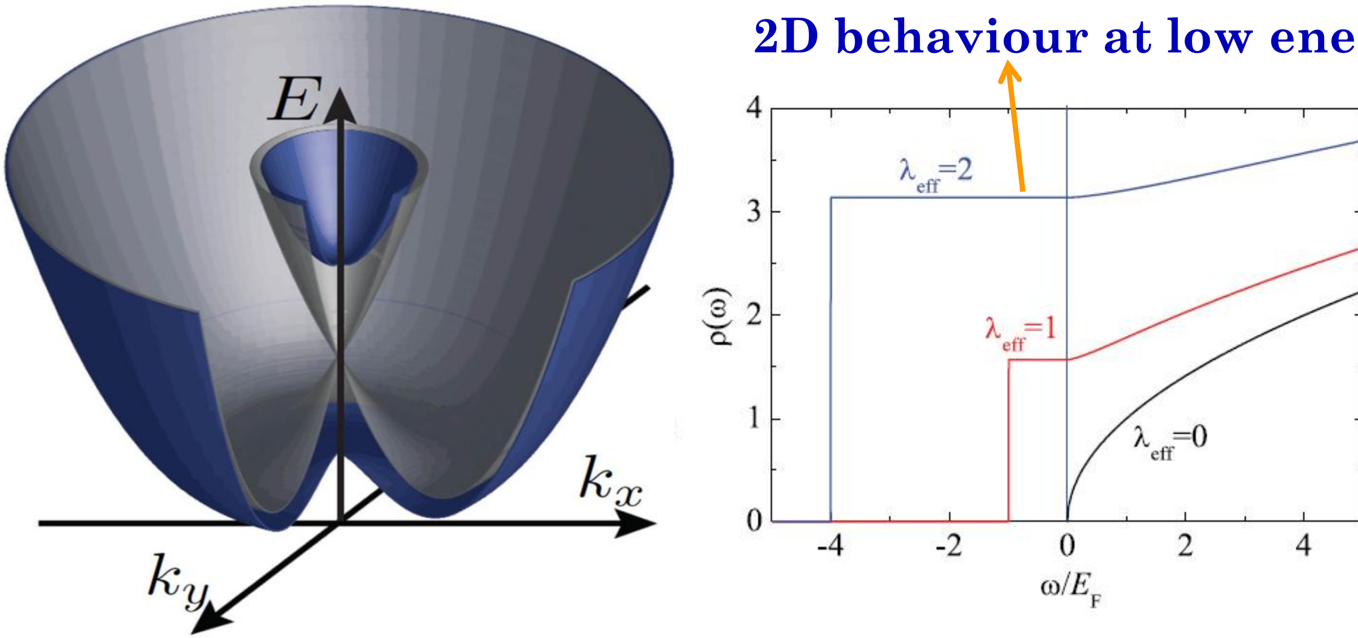
Anyway, I will show you **atomic topological superfluid** in the next lecture.

Single-particle state (Rashba SOC)

Rashba SOC: $V_{\mathrm{SO}} = \lambda_{\mathrm{R}}\left(+k_y\sigma_x - k_x\sigma_y\right)$ to be realized yet…

$$E_{\mathbf{k}\pm} = \frac{\mathbf{k}^2}{2m} \pm \sqrt{\lambda^2(k_x^2 + k_y^2) + h^2}.$$

2D behaviour at low energy?!



Left panel: schematic of the single-particle spectrum in the $k_x - k_y$ plane. A energy gap opens at $k = 0$, due to a non-zero out-of-plane Zeeman field h. Right panel: density of states of a 3D homogeneous Rashba spin–orbit coupled system at several SOC strengths, in units of mk_F.

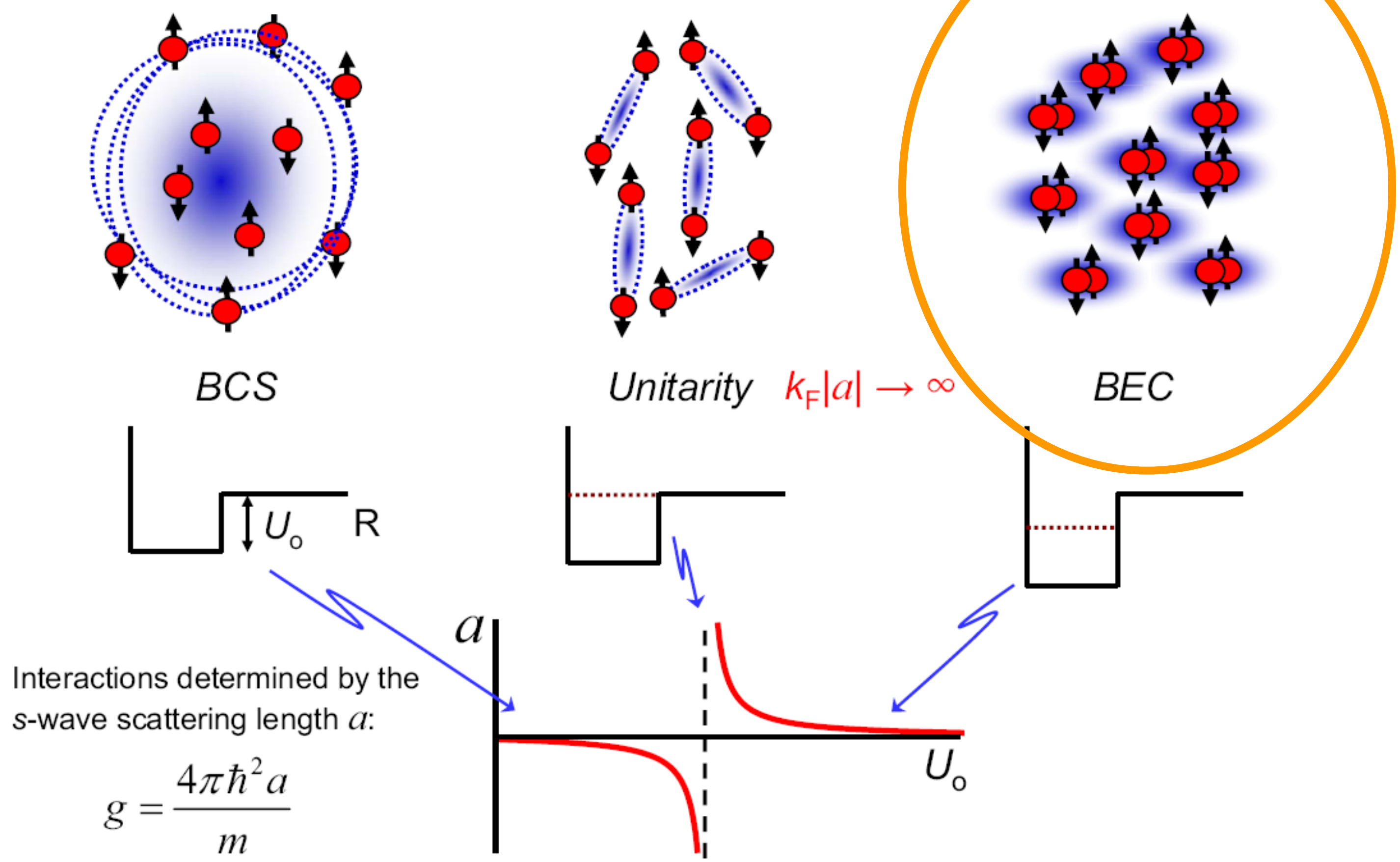
Two interacting atoms with spin-orbit coupling

Two-particle bound state

Let us consider the inter-atomic interactions:

$$\mathcal{H}_{int} = U_0 \int d\mathbf{r}\psi_\uparrow^\dagger(\mathbf{r})\,\psi_\downarrow^\dagger(\mathbf{r})\,\psi_\downarrow(\mathbf{r})\,\psi_\uparrow(\mathbf{r}) \quad \text{or} \quad \mathcal{H}_{int} = U_0 \int d\mathbf{r}\tilde{\psi}_\uparrow^\dagger(\mathbf{r})\,\tilde{\psi}_\downarrow^\dagger(\mathbf{r})\,\tilde{\psi}_\downarrow(\mathbf{r})\,\tilde{\psi}_\uparrow(\mathbf{r})$$

BCS pairing crosses over to Bose-Einstein condensation of molecules with increasing U_o:



3D BEC-BCS crossover without SOC: singlet pairing

Two-particle bound state with ERD-SOC

$$\frac{\hbar^2\left(k_R^2+\mathbf{k}^2\right)}{2M}+h\sigma_x+\lambda k_x\sigma_z \qquad \delta=0$$

Solving: $$\left(\mathcal{H}_0+\mathcal{H}_{\text{int}}\right)|\Phi_{2B}\rangle=E_0|\Phi_{2B}\rangle$$

In the presence of spin-orbit coupling, the wave-function of initial two-particle bound state has both spin singlet and triplet components. The wave-function at zero center-mass momentum, $|\Phi_{2B}\rangle$, may be written as,

$$|\Phi_{2B}\rangle=\frac{1}{\sqrt{\mathcal{C}}}\sum_{\mathbf{k}}\left[\psi_{\uparrow\downarrow}(\mathbf{k})\,c_{\mathbf{k}\uparrow}^{\dagger}c_{-\mathbf{k}\downarrow}^{\dagger}+\psi_{\downarrow\uparrow}(\mathbf{k})\,c_{\mathbf{k}\downarrow}^{\dagger}c_{-\mathbf{k}\uparrow}^{\dagger}+\psi_{\uparrow\uparrow}(\mathbf{k})\,c_{\mathbf{k}\uparrow}^{\dagger}c_{-\mathbf{k}\uparrow}^{\dagger}+\psi_{\downarrow\downarrow}(\mathbf{k})\,c_{\mathbf{k}\downarrow}^{\dagger}c_{-\mathbf{k}\downarrow}^{\dagger}\right]|\text{vac}\rangle,$$

where $c_{\mathbf{k}\uparrow}^{\dagger}$ and $c_{\mathbf{k}\downarrow}^{\dagger}$ are creation field operators of spin-up and spin-down atoms with momentum $\mathbf{k}$ and $\mathcal{C}\equiv\sum_{\mathbf{k}}[|\psi_s(\mathbf{k})|^2+|\psi_a(\mathbf{k})|^2+|\psi_{\uparrow\uparrow}(\mathbf{k})|^2+|\psi_{\downarrow\downarrow}(\mathbf{k})|^2]$ is the normalization factor for the two-particle wave-function. With a contact interaction with bare interaction strength U_0, the Schrödinger equation for the two-particle wavefunction takes the form,

$$\left[E_0-\left(\frac{\hbar^2k_R^2}{m}+\frac{\hbar^2k^2}{m}+2\lambda k_x\right)\right]\psi_{\uparrow\downarrow}(\mathbf{k})=+\frac{U_0}{2}\sum_{\mathbf{k}'}[\psi_{\uparrow\downarrow}(\mathbf{k}')-\psi_{\downarrow\uparrow}(\mathbf{k}')]+h\psi_{\uparrow\uparrow}(\mathbf{k})+h\psi_{\downarrow\downarrow}(\mathbf{k}),$$

$$\left[E_0-\left(\frac{\hbar^2k_R^2}{m}+\frac{\hbar^2k^2}{m}-2\lambda k_x\right)\right]\psi_{\downarrow\uparrow}(\mathbf{k})=-\frac{U_0}{2}\sum_{\mathbf{k}'}[\psi_{\uparrow\downarrow}(\mathbf{k}')-\psi_{\downarrow\uparrow}(\mathbf{k}')]+h\psi_{\uparrow\uparrow}(\mathbf{k})+h\psi_{\downarrow\downarrow}(\mathbf{k}),$$

$$\left[E_0-\left(\frac{\hbar^2k_R^2}{m}+\frac{\hbar^2k^2}{m}\right)\right]\psi_{\uparrow\uparrow}(\mathbf{k})=h\psi_{\uparrow\downarrow}(\mathbf{k})+h\psi_{\downarrow\uparrow}(\mathbf{k}),$$

$$\left[E_0-\left(\frac{\hbar^2k_R^2}{m}+\frac{\hbar^2k^2}{m}\right)\right]\psi_{\downarrow\downarrow}(\mathbf{k})=h\psi_{\uparrow\downarrow}(\mathbf{k})+h\psi_{\downarrow\uparrow}(\mathbf{k}),$$

Hu *et al.*, *PRA* **86**, 053627 (2012).

Two-particle bound state with ERD-SOC

Defining:

$$A_{\mathbf{k}} \equiv E_0 - (\hbar^2 k_R^2/m + \hbar^2 k^2/m)$$

$$\psi_s(\mathbf{k}) = \frac{1}{\sqrt{2}}\left[\psi_{\uparrow\downarrow}(\mathbf{k}) - \psi_{\downarrow\uparrow}(\mathbf{k})\right]$$

$$\psi_a(\mathbf{k}) = \frac{1}{\sqrt{2}}\left[\psi_{\uparrow\downarrow}(\mathbf{k}) + \psi_{\downarrow\uparrow}(\mathbf{k})\right]$$

Wavefunctions:

$$\psi_s(\mathbf{k}) = \frac{1}{h^2 + \lambda^2 k_x^2}\left[\frac{h^2}{A_{\mathbf{k}}} + \frac{\lambda^2 k_x^2 A_{\mathbf{k}}}{A_{\mathbf{k}}^2 - 4(h^2 + \lambda^2 k_x^2)}\right]$$

$$\psi_a(\mathbf{k}) = \lambda k_x \left[\frac{1}{A_{\mathbf{k}} - 2h} + \frac{1}{A_{\mathbf{k}} + 2h}\right]\psi_s(\mathbf{k})$$

$$\psi_{\uparrow\uparrow}(\mathbf{k}) = \frac{\sqrt{2}h}{A_{\mathbf{k}}}\psi_a(\mathbf{k})$$

$$\psi_{\downarrow\downarrow}(\mathbf{k}) = \frac{\sqrt{2}h}{A_{\mathbf{k}}}\psi_a(\mathbf{k})$$

$$\frac{m}{4\pi\hbar^2 a} = \frac{1}{U_0} + \sum_{\mathbf{k}} \frac{m}{\hbar^2 \mathbf{k}^2}$$ is used for U_0

Equation for energy:

$$\frac{m}{4\pi\hbar^2 a_s} - \sum_{\mathbf{k}}\left[\psi_s(\mathbf{k}) + \frac{m}{\hbar^2 k^2}\right] = 0.$$

Two-particle bound state with ERD-SOC

If no SOC, then $\psi_s(k) = \dfrac{1}{E_0 - (\hbar k)^2/m}$

Equation for energy: $$\frac{m}{4\pi\hbar^2 a_s} - \sum_{\mathbf{k}} \left[\psi_s(\mathbf{k}) + \frac{m}{\hbar^2 k^2} \right] = 0.$$

$$\downarrow$$

$E_0 = -\dfrac{\hbar^2}{m{a^2}_s}$ and $\psi_s(r) \propto e^{-r/a_s}$

Two-particle bound state with a general SOC

for the most general form of SOC,

$$V_{\mathrm{SO}}(\hat{\mathbf{k}}) = \sum_{i=x,y,z} (\lambda_i \hat{k}_i + h_i)\hat{\sigma}_i,$$

where λ_i is the strength of SOC in the direction $i = (x, y, z)$ and h_i denotes the effective Zeeman field. The eigenenergy $E(\mathbf{q})$ of a two-body eigenstate with momentum $\mathbf{q}$ satisfies the equation:

$$\frac{m}{4\pi a_s} = \frac{1}{V}\sum_{\mathbf{k}}\left[\left(\mathcal{E}_{\mathbf{k},\mathbf{q}} - \frac{4\mathcal{E}_{\mathbf{k},\mathbf{q}}^2(\boldsymbol{\lambda}\cdot\mathbf{k})^2 - 4\left[\sum_{i=x,y,z}\lambda_i k_i(\lambda_i q_i + 2h_i)\right]^2}{\mathcal{E}_{\mathbf{k},\mathbf{q}}[\mathcal{E}_{\mathbf{k},\mathbf{q}}^2 - \sum_{i=x,y,z}(\lambda_i q_i + 2h_i)^2]}\right)^{-1} + \frac{1}{2\epsilon_{\mathbf{k}}}\right],$$

where $\mathcal{E}_{\mathbf{k},\mathbf{q}} \equiv E(\mathbf{q}) - \epsilon_{\frac{\mathbf{q}}{2}+\mathbf{k}} - \epsilon_{\frac{\mathbf{q}}{2}-\mathbf{k}}$ and $\epsilon_{\mathbf{k}} = k^2/(2m)$.

L. Dong, L. Jiang, HH, & H. Pu, *Phys. Rev. A* **87**, 043616 (2013).

Two-particle bound state with a general SOC

The pairs may have an effective mass larger than $2m$.

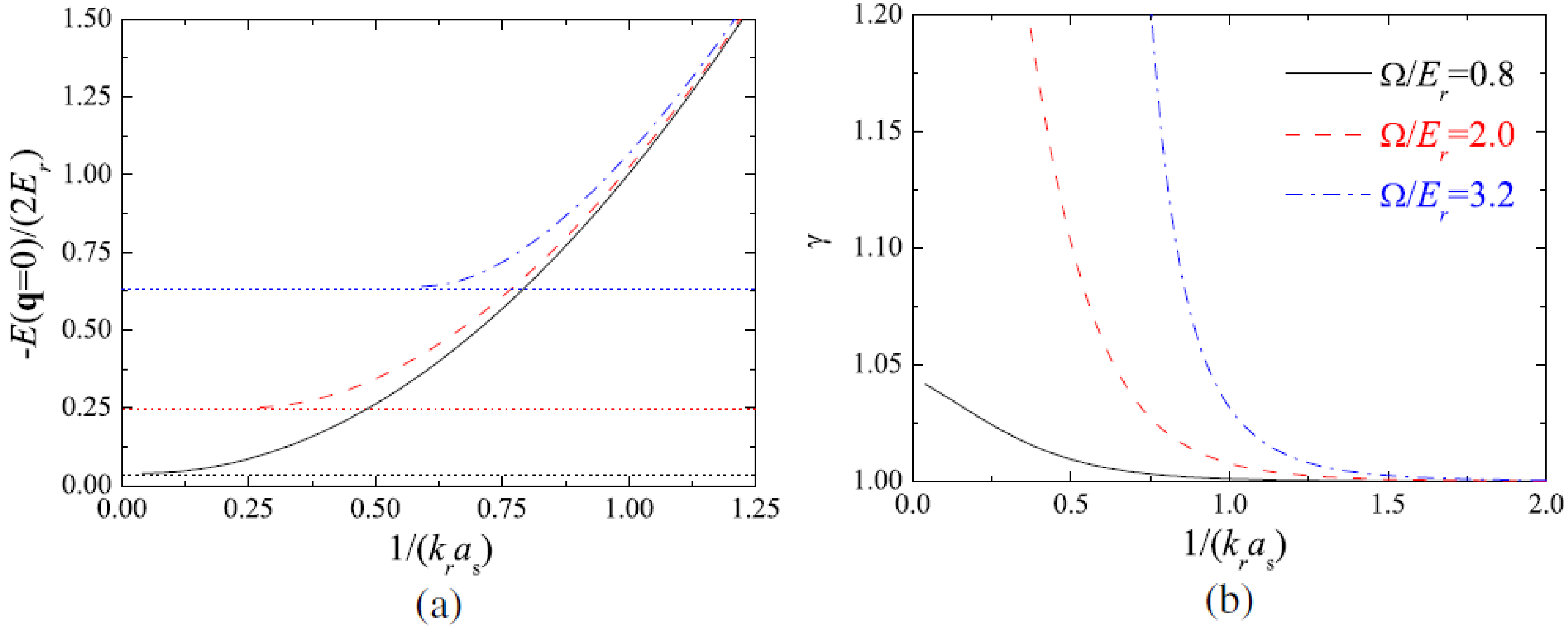
For example, for the bound state with zero center-of-mass momentum $\mathbf{q} = 0$, it would have a quadratic dispersion for small $\mathbf{p}$,

$$E(\mathbf{p}) = E(\mathbf{0}) + \frac{p_x^2}{2M_x} + \frac{p_y^2}{2M_y} + \frac{p_z^2}{2M_z}.$$

The effective mass of the bound state M_i ($i = x, y, z$) can then be determined directly from this dispersion relation.

Two-particle bound state with ERD-SOC

δ=0



Energy $-E(q=0)$ (a) and effective mass ratio $\gamma = M_x/(2m)$ (b) of the two-particle ground bound state in the presence of 1D equal-weight Rashba–Dresselhaus SOC, at zero detuning $\delta = 0$ and at three coupling strengths of Raman beams: $\Omega = 0.8E_r$ (solid line), $2E_r$ (dashed line), and $3.2E_r$ (dot-dashed line). The horizontal dotted lines in (a) correspond to the threshold energies $-2E_{\min}$ where the bound states disappear.

Two features: (i) bound state at $a_s > 0$ only and ERD SOC does not favour two-body bound state; (ii) In the axis of SOC, pair mass > $2m$.

Two-particle bound state (rf-spectroscopy)

Franck-Condon factor (Fermi golden rule again):

$$F(\omega) = \left|\langle \Phi_f | \mathcal{V}_{rf} | \Phi_{2B} \rangle\right|^2 \delta\left[\omega - \omega_{3\downarrow} - \frac{E_f - E_0}{\hbar}\right]$$

final state energy E_f

Initial state energy of the two-particle bound state

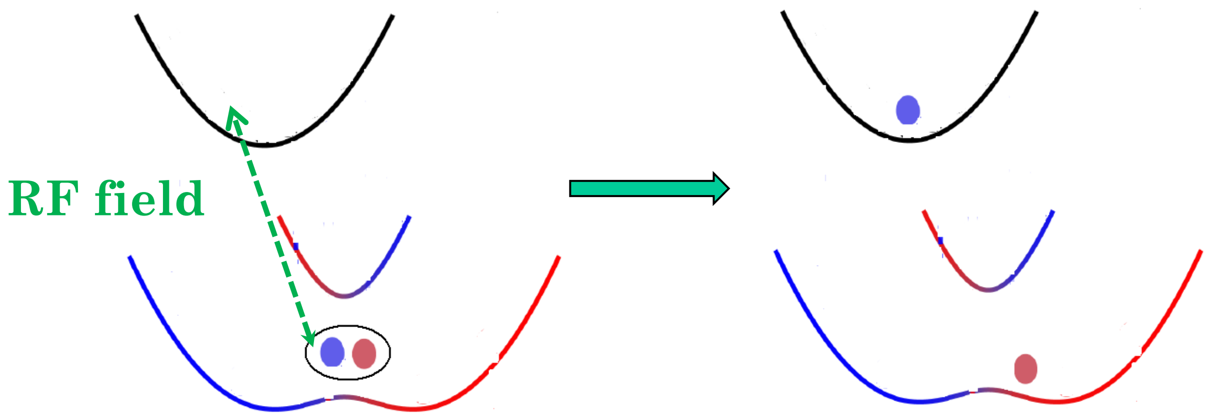
Two-particle bound state (rf-spectroscopy)

Momentum-resolved rf transfer strength:

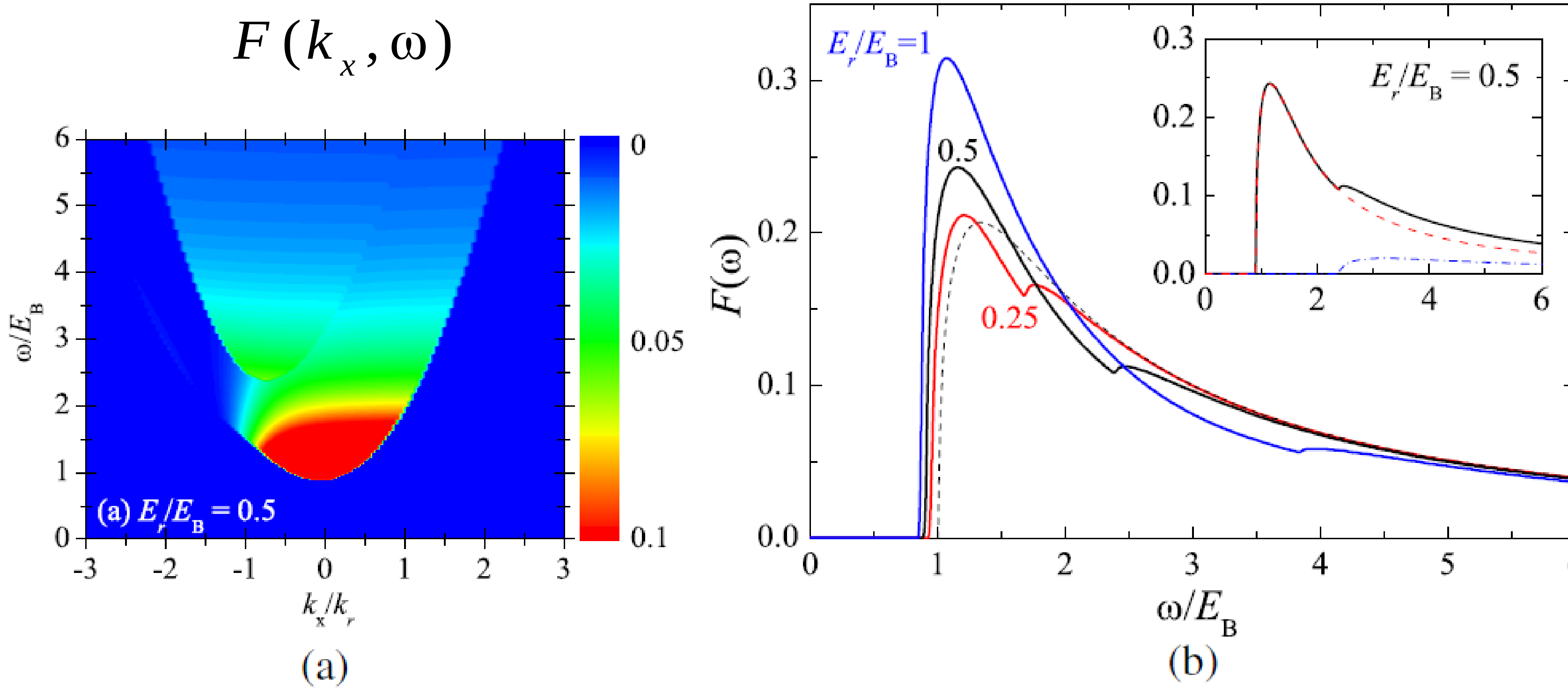
$$F\left(k_x,\omega\right)=\frac{1}{\mathcal{C}}\sum_{\mathbf{q}_\perp}\left[s_{\mathbf{q}+}^2\delta\left(\omega-\frac{\mathcal{E}_{\mathbf{q}+}}{\hbar}\right)+s_{\mathbf{q}-}^2\delta\left(\omega-\frac{\mathcal{E}_{\mathbf{q}-}}{\hbar}\right)\right]$$

$$s_{\mathbf{q}+}=\left[\psi_s\left(\mathbf{q}\right)+\psi_a\left(\mathbf{q}\right)\right]\cos\theta_{\mathbf{q}}+\sqrt{2}\psi_{\downarrow\downarrow}\left(\mathbf{q}\right)\sin\theta_{\mathbf{q}},$$
$$s_{\mathbf{q}-}=\left[\psi_s\left(\mathbf{q}\right)+\psi_a\left(\mathbf{q}\right)\right]\sin\theta_{\mathbf{q}}-\sqrt{2}\psi_{\downarrow\downarrow}\left(\mathbf{q}\right)\cos\theta_{\mathbf{q}}.$$

$$\mathcal{E}_{\mathbf{q}\pm}\equiv\epsilon_B+\frac{\hbar^2\left(k_R^2+q^2\right)}{2m}\pm\sqrt{h^2+\lambda^2q_x^2}+\frac{\hbar^2\left(\mathbf{q}+k_R\mathbf{e}_x\right)^2}{2m}$$



Two-particle bound state (rf-spectroscopy)



(a) Momentum-resolved rf spectroscopy (a) and integrated rf spectroscopy (b) of the two-particle bound state at $\delta = 0$ and $\Omega = 2E_r$. The energy of rf photon ω is measured in units of a binding energy $E_B \equiv 1/(ma_s^2)$ and we have set $\omega_{3\downarrow} = 0$. In the right panel, the dashed line in the main figure plots the rf line-shape in the absence of SOC: $F(\omega) = (2/\pi)\sqrt{\omega - E_B}/\omega^2$. The inset highlights the different contribution from the two final states, as described in the text.

to be observed …

Theoretical framework: functional path integral

The partition function: $\mathcal{Z} = \int \mathcal{D}[\psi(\mathbf{r},\tau),\bar{\psi}(\mathbf{r},\tau)] \exp\{-S[\psi(\mathbf{r},\tau),\bar{\psi}(\mathbf{r},\tau)]\}$

(1) HS transformation

(action) $S[\psi,\bar{\psi}] = \int_0^\beta d\tau \Big[\int d\mathbf{r} \sum_\sigma \bar{\psi}_\sigma(\mathbf{r},\tau)\partial_\tau \psi_\sigma(\mathbf{r},\tau) + \mathcal{H}(\psi,\bar{\psi})\Big]$

$$\mathcal{Z} = \int \mathcal{D}[\Phi,\bar{\Phi};\Delta,\bar{\Delta}] \exp\left\{-\int d\tau \int d\mathbf{r} \int d\tau' \int d\mathbf{r}' \left[-\frac{1}{2}\bar{\Phi}(\mathbf{r},\tau)\mathcal{G}^{-1}\Phi(\mathbf{r}',\tau') - \frac{|\Delta(\mathbf{r},\tau)|^2}{U_0}\delta(\mathbf{r}-\mathbf{r}')\delta(\tau-\tau')\right] - \frac{\beta}{V}\sum_{\mathbf{k}} \xi_{\mathbf{k}}\right\}$$

(2) Integrate out fermionic fields, and expand the pairing field around its mean-field

G_0: Green function of fermions

(Mean-field) $$S_0 = \int_0^\beta d\tau \int d\mathbf{r} \left(-\frac{\Delta_0^2}{U_0}\right) - \frac{1}{2}\mathrm{Tr}\ln[-\mathcal{G}_0^{-1}] + \frac{\beta}{V}\sum_{\mathbf{k}} \xi_{\mathbf{k}}$$

(Pair fluctuations) $$\Delta S = k_B T \frac{1}{V} \sum_{q=\mathbf{q},i\nu_n} [-\Gamma^{-1}(q)]\delta\Delta(q)\delta\bar{\Delta}(q)$$

$\Gamma(\mathbf{q},i\nu_n)$: Green function of Cooper pairs

L. Jiang, X.-J. Liu, HH, & H. Pu, *PRA* **84**, 063618 (2011); Carlos Sa de Melo *et al.* *PRL* (1993).

Two-body study I: bound state with Rashba SOC

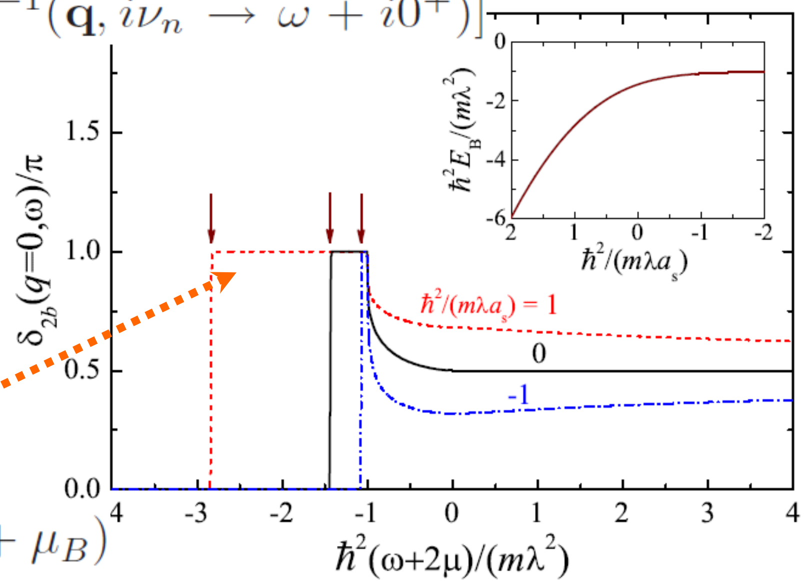
$$\mathcal{H} = \int d\mathbf{r} \left\{ \psi^{+} \left[\xi_{\mathbf{k}} + \lambda(\hat{k}_y \hat{\sigma}_x - \hat{k}_x \hat{\sigma}_y) \right] \psi + U_0 \psi_{\uparrow}^{+} \psi_{\downarrow}^{+} \psi_{\downarrow} \psi_{\uparrow} \right\}$$

Rashba SO coupling, 3D Fermi gas

$$\Delta S = \sum_{\mathbf{q}, i\nu_n} \left[-\Gamma^{-1}(q) \right] \delta\Delta(q) \delta\bar{\Delta}(q),$$ $\Gamma(\mathbf{q},\omega)$: Green function of pairs

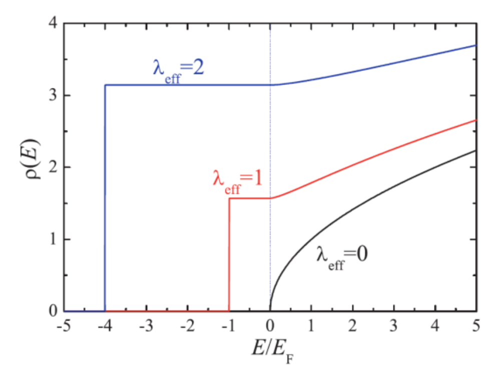
$$\Gamma^{-1}(\mathbf{0}, \omega) = \frac{m}{4\pi\hbar^2 a_s} - \frac{1}{2} \sum_{\mathbf{k}} \left[\sum_{\alpha=\pm} \frac{1 - 2f(E_{\mathbf{k},\alpha})}{\omega + i0^{+} - 2E_{\mathbf{k},\alpha}} + \frac{1}{\epsilon_{\mathbf{k}}} \right], \qquad E_{\mathbf{k},\pm} = \xi_{\mathbf{k}} \pm \lambda k_{\perp}$$

$$\delta(\mathbf{q}, \omega) = -\operatorname{Im} \ln[-\Gamma^{-1}(\mathbf{q}, i\nu_n \to \omega + i0^{+})]$$



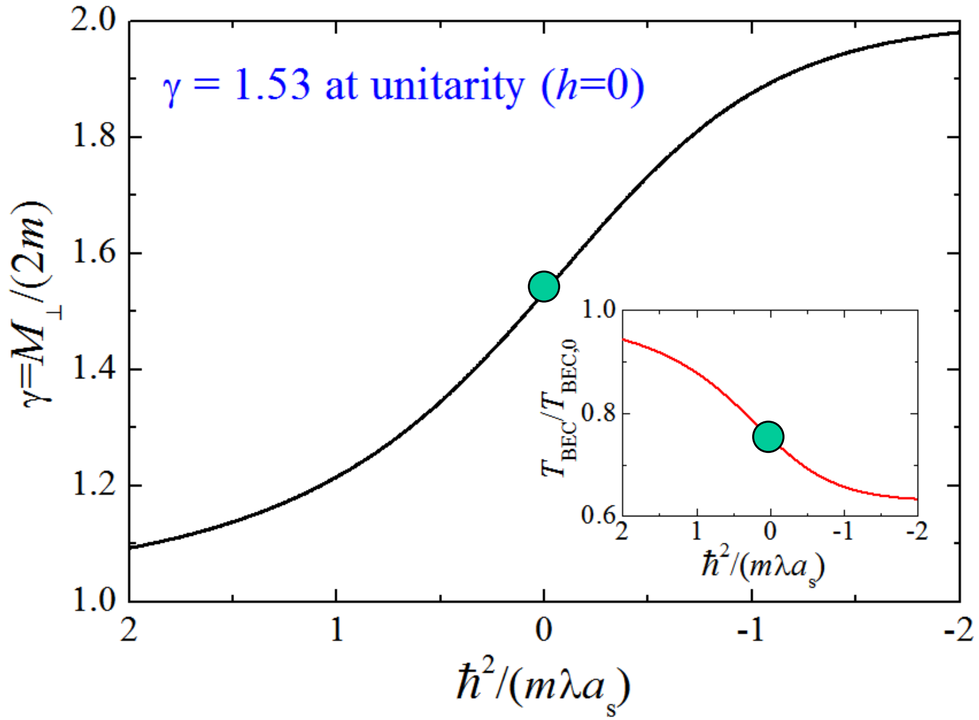
$$\delta_B(\mathbf{q}, \omega) = \pi\Theta(\omega - \epsilon_{\mathbf{q}}^{B} + \mu_B)$$

density of states



Two-body study I: rashbons in the unitary limit

Pairs have **anisotropic mass**: $M_z = 2m$, but

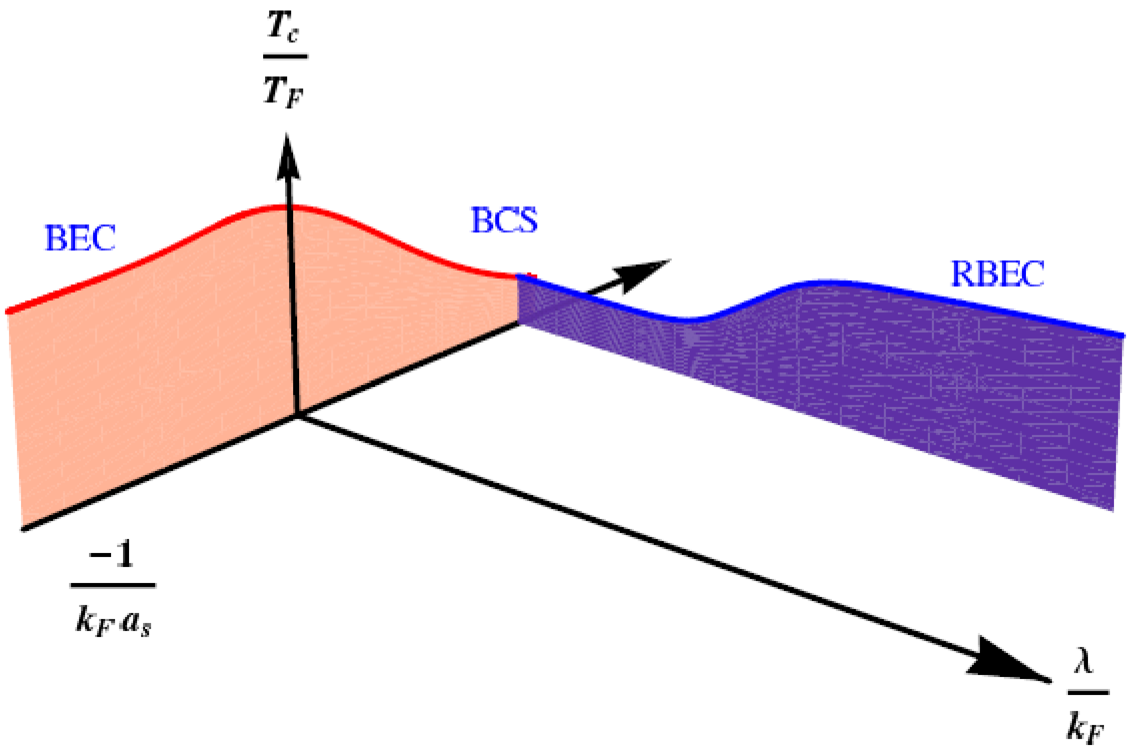


At unitarity, size of rashbons: $a \sim \hbar^2/(m\lambda)$ and the scattering length: $a_B \sim 3\hbar^2/(m\lambda)$.

Rashbons are created by strong Rashba spin-orbit coupling !

HH, L. Jiang, X.-J. Liu, & H. Pu, *Phys. Rev. Lett.* **107**, 195304(2011).

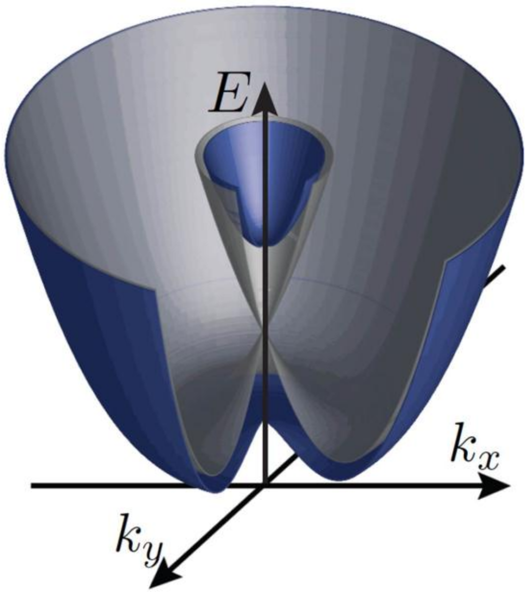
Two-body study I: significance of rashbons



Rashbons are created by strong Rashba spin-orbit coupling !

J. P. Vyasanakere & V. B. Shenoy, *New J. Phys.* 14, 043041 (2012).

Interplay between SOC and interatomic interaction



Problem:

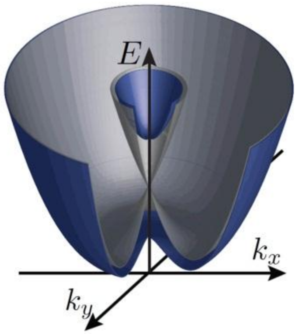
Consider the Rashba spin-orbit coupling, if atoms occupy the low-helicity branch, which may be regarded as a new spin-state, what is the effective interaction between atoms in this new spin-state?

More on (effective) interatomic interactions

Solution: Rewrite the interatomic interaction using the field operator in the helicity representation!

$$\mathcal{H}_s = \begin{bmatrix} \hbar^2 k^2/(2m) + h & \lambda(k_y + ik_x) \\ \lambda(k_y - ik_x) & \hbar^2 k^2/(2m) - h \end{bmatrix},$$

and, by a spin-rotation we obtain the single-particle spectrum,



$$\epsilon_{\mathbf{k}\alpha} = \hbar^2 k^2/(2m) + \alpha\sqrt{h^2 + \lambda^2\left(k_x^2 + k_y^2\right)},$$

where $\alpha = +, -$ denotes the different branch (helicity) of spectrum.

Consider now the spin-rotation. For the upper branch ($\alpha = +$), we need to solve,

$$\begin{bmatrix} +h - \sqrt{h^2 + \lambda^2\left(k_x^2 + k_y^2\right)} & i\lambda(k_x - ik_y) \\ -i\lambda(k_x + ik_y) & -h - \sqrt{h^2 + \lambda^2\left(k_x^2 + k_y^2\right)} \end{bmatrix} \begin{bmatrix} u_+(\mathbf{k}) \\ v_+(\mathbf{k}) \end{bmatrix} = 0.$$

More on (effective) interatomic interactions

Let us define two angles:

$$\boxed{\begin{aligned}\phi_{\mathbf{k}} &= \arccos \frac{k_x}{k_\perp}, \\ \theta_{\mathbf{k}} &= \arctan \left[\sqrt{\left(\frac{h}{\lambda k_\perp}\right)^2 + 1} - \frac{h}{\lambda k_\perp} \right],\end{aligned}}$$

where $k_\perp = \sqrt{k_x^2 + k_y^2}$. It is easy to see that, $u_+(\mathbf{k}) = \cos\theta_{\mathbf{k}}$ and $v_+(\mathbf{k}) = -i\sin\theta_{\mathbf{k}} e^{+i\phi_{\mathbf{k}}}$. We find similarly that, for the lower branch ($\alpha = -$), $u_-(\mathbf{k}) = -i\sin\theta_{\mathbf{k}} e^{-i\phi_{\mathbf{k}}}$ and $v_-(\mathbf{k}) = \cos\theta_{\mathbf{k}}$.

Thus, we have,

$$\begin{pmatrix} |\mathbf{k}+\rangle \\ |\mathbf{k}-\rangle \end{pmatrix} = \begin{bmatrix} \cos\theta_{\mathbf{k}} & -i\sin\theta_{\mathbf{k}} e^{+i\phi_{\mathbf{k}}} \\ -i\sin\theta_{\mathbf{k}} e^{-i\phi_{\mathbf{k}}} & \cos\theta_{\mathbf{k}} \end{bmatrix} \begin{pmatrix} |\mathbf{k}\uparrow\rangle \\ |\mathbf{k}\downarrow\rangle \end{pmatrix},$$

or alternatively,

$$\boxed{\begin{pmatrix} |\mathbf{k}\uparrow\rangle \\ |\mathbf{k}\downarrow\rangle \end{pmatrix} = \begin{bmatrix} \cos\theta_{\mathbf{k}} & i\sin\theta_{\mathbf{k}} e^{+i\phi_{\mathbf{k}}} \\ i\sin\theta_{\mathbf{k}} e^{-i\phi_{\mathbf{k}}} & \cos\theta_{\mathbf{k}} \end{bmatrix} \begin{pmatrix} |\mathbf{k}+\rangle \\ |\mathbf{k}-\rangle \end{pmatrix}.}$$

More on (effective) interatomic interactions

$$\boxed{\text{what is the interaction Hamiltonian in the helicity basis?}}$$

In general, we would have some very complicated interaction terms after the spin-rotation.

For example, for the spin-rotation (i.e., Rashba SO),

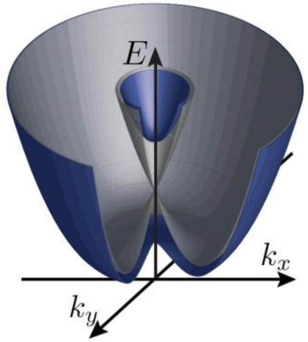
$$\begin{pmatrix} |\mathbf{k}\uparrow\rangle \\ |\mathbf{k}\downarrow\rangle \end{pmatrix} = \begin{bmatrix} \cos\theta_{\mathbf{k}} & i\sin\theta_{\mathbf{k}} e^{+i\phi_{\mathbf{k}}} \\ i\sin\theta_{\mathbf{k}} e^{-i\phi_{\mathbf{k}}} & \cos\theta_{\mathbf{k}} \end{bmatrix} \begin{pmatrix} |\mathbf{k}+\rangle \\ |\mathbf{k}-\rangle \end{pmatrix},$$

the interaction term $\mathcal{H}_{int} = U_0 \sum_{\mathbf{k},\mathbf{k}',\mathbf{q}} \psi^+_{\mathbf{k},\uparrow} \psi^+_{\mathbf{q}-\mathbf{k},\downarrow} \psi_{\mathbf{q}-\mathbf{k}',\downarrow} \psi_{\mathbf{k}',\uparrow}$ is given by,

$$\begin{aligned} & U_0 \sum_{\mathbf{k},\mathbf{k}',\mathbf{q}} \left[\cos\theta_{\mathbf{k}} \psi^+_{\mathbf{k},+} - i\sin\theta_{\mathbf{k}} e^{-i\phi_{\mathbf{k}}} \psi^+_{\mathbf{k},-}\right] \left[-i\sin\theta_{\mathbf{q}-\mathbf{k}} e^{i\phi_{\mathbf{q}-\mathbf{k}}} \psi^+_{\mathbf{q}-\mathbf{k},+} + \cos\theta_{\mathbf{q}-\mathbf{k}} \psi^+_{\mathbf{q}-\mathbf{k},-}\right] \\ & \times \left[i\sin\theta_{\mathbf{q}-\mathbf{k}'} e^{-i\phi_{\mathbf{q}-\mathbf{k}'}} \psi_{\mathbf{q}-\mathbf{k}',+} + \cos\theta_{\mathbf{q}-\mathbf{k}'} \psi_{\mathbf{q}-\mathbf{k}',-}\right] \left[\cos\theta_{\mathbf{k}'} \psi_{\mathbf{k}',+} + i\sin\theta_{\mathbf{k}'} e^{i\phi_{\mathbf{k}'}} \psi_{\mathbf{k}',-}\right]. \end{aligned}$$

More on (effective) interatomic interactions

In case of a large Zeeman field:



$$\mathcal{H}_{int}^{eff} \simeq U_0 \sum_{\mathbf{k},\mathbf{k}',\mathbf{q}} \left(\sin\theta_{\mathbf{k}} e^{-i\phi_{\mathbf{k}}} \cos\theta_{\mathbf{q}-\mathbf{k}} \cos\theta_{\mathbf{q}-\mathbf{k}'} \sin\theta_{\mathbf{k}'} e^{i\phi_{\mathbf{k}'}}\right) \psi^{+}_{\mathbf{k},-}\psi^{+}_{\mathbf{q}-\mathbf{k},-}\psi_{\mathbf{q}-\mathbf{k}',-}\psi_{\mathbf{k}',-},$$

$$\simeq U_0 \sum_{\mathbf{k},\mathbf{k}',\mathbf{q}} \left[\cos\theta_{\mathbf{k}} \sin\theta_{\mathbf{k}} \cos\theta_{\mathbf{k}'} \sin\theta_{\mathbf{k}'} e^{-i(\phi_{\mathbf{k}}-\phi_{\mathbf{k}'})}\right] \psi^{+}_{\mathbf{k},-}\psi^{+}_{\mathbf{q}-\mathbf{k},-}\psi_{\mathbf{q}-\mathbf{k}',-}\psi_{\mathbf{k}',-},$$

where in the second line we take $\mathbf{q} = 0$ at $\cos\theta_{\mathbf{q}-\mathbf{k}}$ and $\cos\theta_{\mathbf{q}-\mathbf{k}'}$ to have a well-defined two-body interaction. The angle $\theta_{\mathbf{k}}$ is given by,

$$\theta_{\mathbf{k}} = \arctan\left[\frac{\lambda k_\perp}{\sqrt{h^2+\lambda^2 k_\perp^2}+h}\right] \text{ and the angle } \phi_{\mathbf{k}} \text{ satisfies, } e^{\pm i\phi_{\mathbf{k}}} = \frac{k_x \pm i k_y}{k_\perp}.$$

We then have,

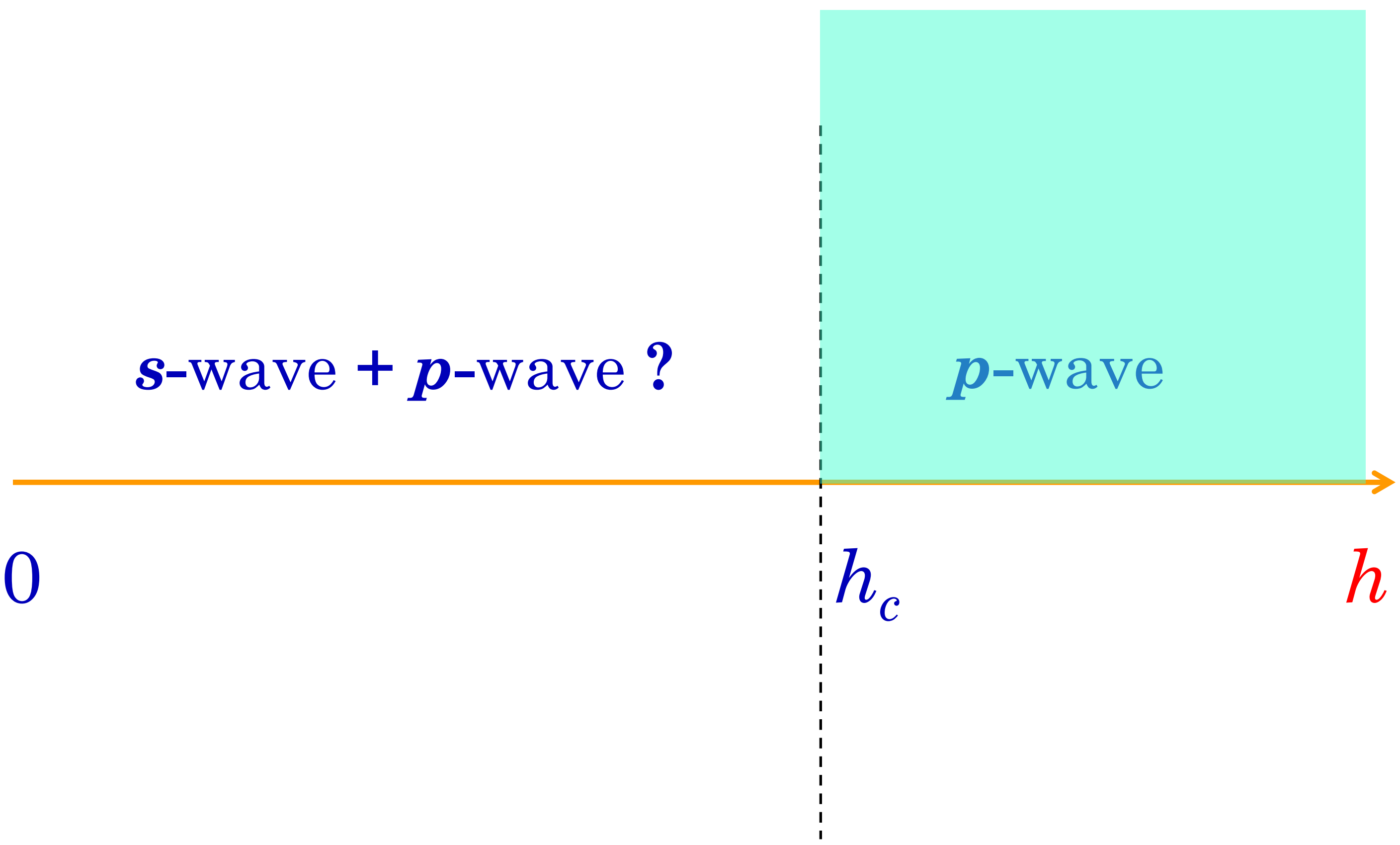
$$\boxed{\mathcal{H}_{int}^{eff} \simeq \sum_{\mathbf{k},\mathbf{k}',\mathbf{q}} V_p\left(\mathbf{k}-\mathbf{k}'\right) \psi^{+}_{\mathbf{k},-}\psi^{+}_{\mathbf{q}-\mathbf{k},-}\psi_{\mathbf{q}-\mathbf{k}',-}\psi_{\mathbf{k}',-},}$$

where

$$\begin{aligned} V_p\left(\mathbf{k}-\mathbf{k}'\right) &= \cos\theta_{\mathbf{k}} \sin\theta_{\mathbf{k}} \cos\theta_{\mathbf{k}'} \sin\theta_{\mathbf{k}'} e^{-i(\phi_{\mathbf{k}}-\phi_{\mathbf{k}'})}, \\ &= \frac{U_0}{4} \frac{(k_x - i k_y)\left(k'_x + i k'_y\right)}{\sqrt{(h/\lambda)^2 + (k_\perp)^2}\sqrt{(h/\lambda)^2 + \left(k'_\perp\right)^2}}, \end{aligned}$$

is the effective p-wave interaction between fermions in the lower branch

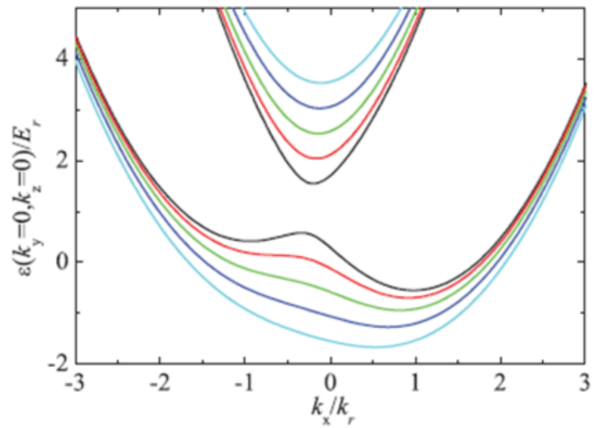
More on (effective) interatomic interactions



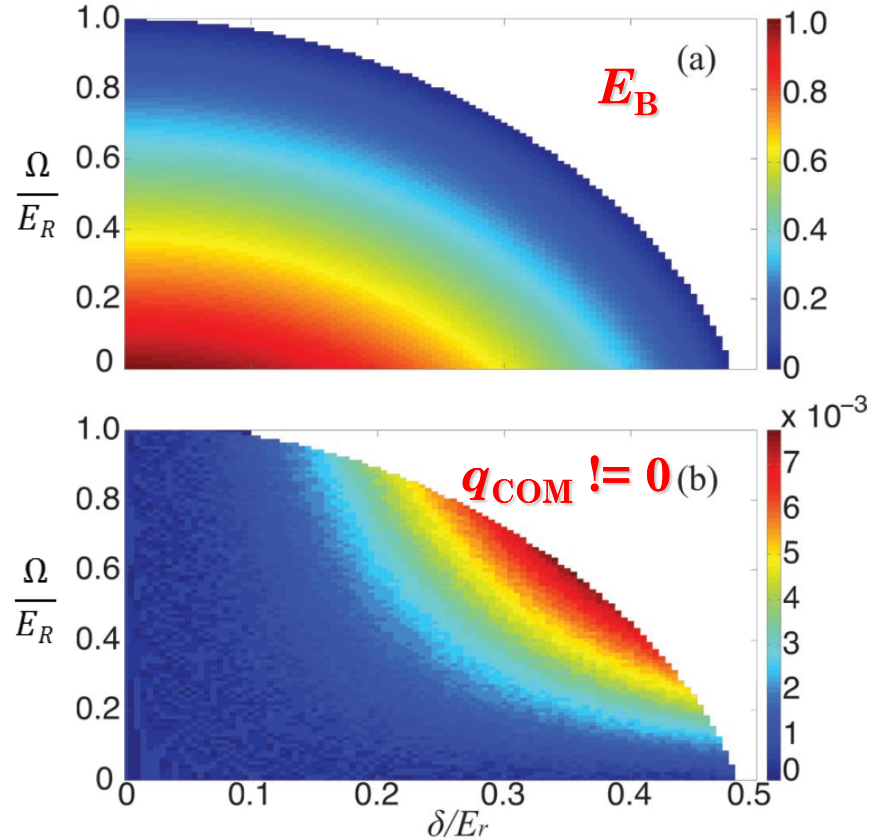
In our nature, no *p*-wave superconductors found so far !!!

Two-body study II: bound state with finite q_{COM}

$$H = \frac{\hbar^2 \hat{k}^2}{2m} + \lambda_{SO} k_x \sigma_y + \frac{\delta}{2}\sigma_y + \frac{\Omega}{2}\sigma_z$$



single-particle spectrum



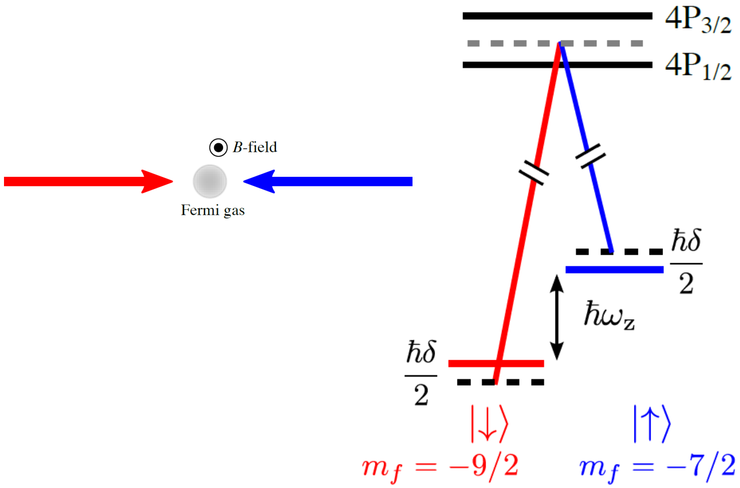
two-particle spectrum

L. Dong, L. Jiang, HH, & H. Pu, *Phys. Rev. A* **87**, 043616 (2013).

The significance of finite q_{COM}:

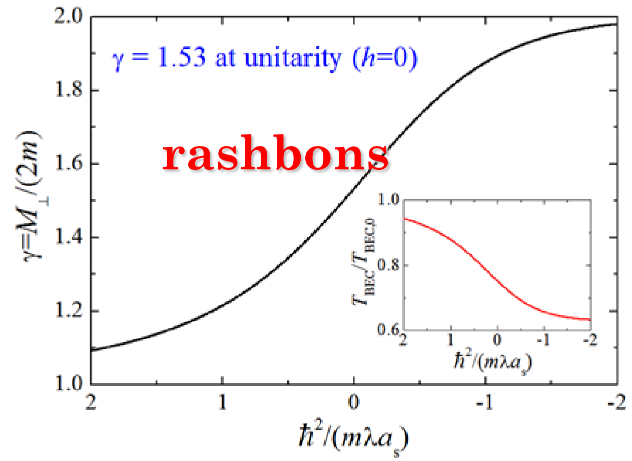
- **Implying inhomogeneous Fulde-Ferrell pairing, to be detailed later;**
- **q_{COM} is along the direction of SOC;**
- **The magnitude of q_{COM} can be greatly enlarged by many-body effect.**

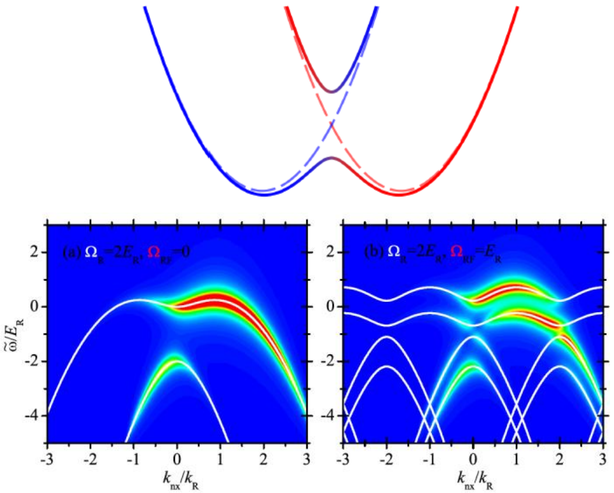
Summary



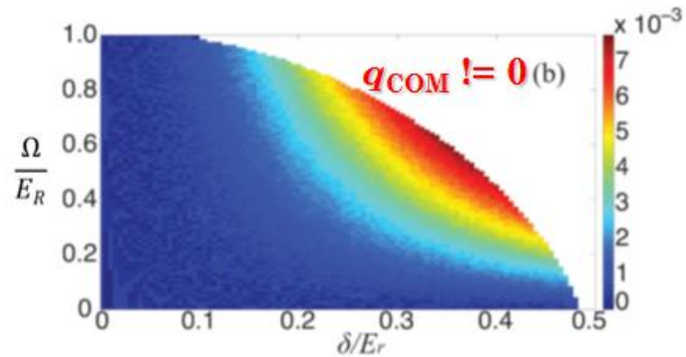
synthetic spin-orbit coupling

two-particle bound state





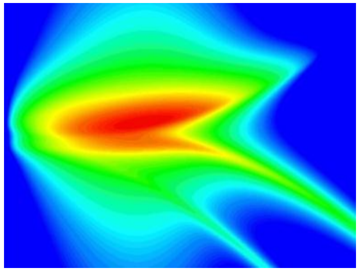
single-particle spectrum



• Experimental realization of SOC and two-body study (I & II)

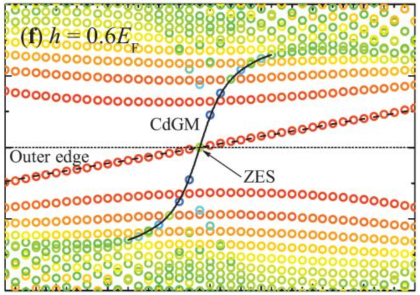
(No Zeeman field, **two-body I**)

• **Anisotropic superfluidity of rashbons**



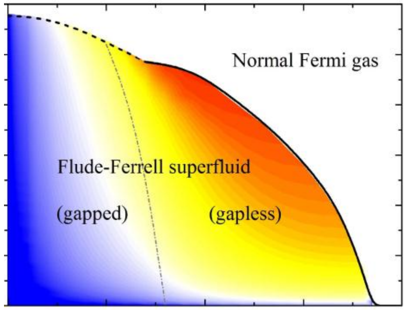
(Out-of-plane B-field, ***p*-wave pairing**)

• **Topological superfluid and Majorana fermions**



(In-plane B-field, **two-body II**)

• **Fulde-Ferrell superfluidity**



Lecture II: many-body physics, mean-field

I. MODEL HAMILTONIAN

Let us start from the model Hamiltonian for a 3D Fermi gas with 3D Rashba spin-orbit coupling $\lambda(\sigma_x\hat{k}_x+\sigma_y\hat{k}_y+\sigma_z\hat{k}_z)$ and a magnetic field h along z-direction, $\mathcal{H}=\mathcal{H}_0+\mathcal{H}_{int}$, where

$$\mathcal{H}_0=\int d\mathbf{x}\left[\psi_\uparrow^\dagger(\mathbf{x}),\psi_\downarrow^\dagger(\mathbf{x})\right]\left[\begin{array}{cc}\hat{\xi}_\mathbf{k}+\lambda\hat{k}_z-h & \lambda(\hat{k}_x-i\hat{k}_y)\\ \lambda(\hat{k}_x+i\hat{k}_y) & \hat{\xi}_\mathbf{k}-\lambda\hat{k}_z+h\end{array}\right]\left[\begin{array}{c}\psi_\uparrow(\mathbf{x})\\ \psi_\downarrow(\mathbf{x})\end{array}\right] \qquad (1)$$

and the interaction Hamiltonian is,

$$\mathcal{H}_{int}=U_0\int d\mathbf{x}\psi_\uparrow^\dagger(\mathbf{x})\,\psi_\downarrow^\dagger(\mathbf{x})\,\psi_\downarrow(\mathbf{x})\,\psi_\uparrow(\mathbf{x})\,. \qquad (2)$$

Here, we have defined $\hat{\xi}_\mathbf{k}\equiv-\hbar^2\nabla^2/(2m)-\mu$, $\hat{k}_x=-i\partial_x$, $\hat{k}_y=-i\partial_y$, and $\hat{k}_z=-i\partial_z$.

Mean-field theory : general treatment

II. MEAN-FIELD BDG THEORY

According to Lin's two-body calculation, let us assume a FF-like order parameter $\Delta(\mathbf{x}) = -U_0 \langle \psi_\downarrow(\mathbf{x}) \psi_\uparrow(\mathbf{x}) \rangle = \Delta \exp[iqz]$ along the z-axis and consider the mean-field decoupling,

$$\mathcal{H}_{int} \simeq -\int d\mathbf{x} \left[\Delta(\mathbf{x}) \psi_\uparrow^\dagger(\mathbf{x}) \psi_\downarrow^\dagger(\mathbf{x}) + \mathrm{H.c.} \right] - \frac{1}{U_0} \int d\mathbf{x} \left| \Delta(\mathbf{x}) \right|^2 . \tag{3}$$

Within this mean-field BdG theory, the total Hamiltonian can be written into the form,

$$\mathcal{H}_{MF} = \frac{1}{2} \int d\mathbf{x} \Phi^\dagger(\mathbf{x}) \mathcal{H}_{BdG} \Phi(\mathbf{x}) - \frac{\Delta^2}{U_0} V + \sum_{\mathbf{k}} \xi_{\mathbf{k}}, \tag{4}$$

where $\Phi(\mathbf{x}) \equiv [\psi_\uparrow(\mathbf{x}), \psi_\downarrow(\mathbf{x}), \psi_\uparrow^\dagger(\mathbf{x}), \psi_\downarrow^\dagger(\mathbf{x})]^T$ is a Nambu spinor, and

$$\mathcal{H}_{BdG} \equiv \begin{bmatrix} \hat{\xi}_{\mathbf{k}} + \lambda \hat{k}_z - h & \lambda(\hat{k}_x - i\hat{k}_y) & 0 & -\Delta(\mathbf{x}) \\ \lambda(\hat{k}_x + i\hat{k}_y) & \hat{\xi}_{\mathbf{k}} - \lambda \hat{k}_z + h & \Delta(\mathbf{x}) & 0 \\ 0 & \Delta^*(\mathbf{x}) & -\hat{\xi}_{\mathbf{k}} + \lambda \hat{k}_z + h & \lambda(\hat{k}_x + i\hat{k}_y) \\ -\Delta^*(\mathbf{x}) & 0 & \lambda(\hat{k}_x - i\hat{k}_y) & -\hat{\xi}_{\mathbf{k}} - \lambda \hat{k}_z - h \end{bmatrix}. \tag{5}$$

Bogoliubov transformation

$$\boxed{\begin{aligned}\hat{b} &= u\hat{a} + v\hat{a}^\dagger \\ \hat{b}^\dagger &= u^*\hat{a}^\dagger + v^*\hat{a}\end{aligned}}$$

$$\Phi^+ H_{BdG} \Phi$$

$$\downarrow$$

$$\Phi^+ X^+ \; E \; X\Phi$$

field operators for Bogoliubov quasiparticles

A. Bogoliubov quasiparticles for the Hamiltonian $\mathcal{H}_{BdG}$

Now, let us turn to solve the Bogoliubov equation,

$$\mathcal{H}_{BdG}\Phi_{\mathbf{k}}(\mathbf{x}) = E_{\mathbf{k}}\Phi_{\mathbf{k}}(\mathbf{x}), \tag{6}$$

where

$$\boxed{\Delta(\boldsymbol{x}) = \Delta_0 e^{iqz}}$$

$$\Phi_{\mathbf{k}}(\mathbf{x}) \equiv \begin{bmatrix} u_{\mathbf{k}\uparrow}e^{+iqz/2} \\ u_{\mathbf{k}\downarrow}e^{+iqz/2} \\ v_{\mathbf{k}\uparrow}e^{-iqz/2} \\ v_{\mathbf{k}\downarrow}e^{-iqz/2} \end{bmatrix} e^{i\mathbf{kx}} \tag{7}$$

and $E_{\mathbf{k}}$ are the wave-function and energy of the Bogoliubov quasiparticles, respectively. Therefore, we will have,

$$[\mathcal{H}_{BdG}] \begin{bmatrix} u_{\mathbf{k}\uparrow} \\ u_{\mathbf{k}\downarrow} \\ v_{\mathbf{k}\uparrow} \\ v_{\mathbf{k}\downarrow} \end{bmatrix} = E_{\mathbf{k}} \begin{bmatrix} u_{\mathbf{k}\uparrow} \\ u_{\mathbf{k}\downarrow} \\ v_{\mathbf{k}\uparrow} \\ v_{\mathbf{k}\downarrow} \end{bmatrix}, \tag{8}$$

where $[\mathcal{H}_{BdG}]$ is given by,

Mean-field theory : general treatment

$$\begin{bmatrix} \xi_{\mathbf{k}+\frac{q}{2}\mathbf{e}_z} + \lambda\left(k_z + \frac{q}{2}\right) - h & \lambda\left(k_x - ik_y\right) & 0 & -\Delta \\ \lambda\left(k_x + ik_y\right) & \xi_{\mathbf{k}+\frac{q}{2}\mathbf{e}_z} - \lambda\left(k_z + \frac{q}{2}\right) + h & \Delta & 0 \\ 0 & \Delta & -\xi_{\mathbf{k}-\frac{q}{2}\mathbf{e}_z} + \lambda\left(k_z - \frac{q}{2}\right) + h & \lambda\left(k_x + ik_y\right) \\ -\Delta & 0 & \lambda\left(k_x - ik_y\right) & -\xi_{\mathbf{k}-\frac{q}{2}\mathbf{e}_z} - \lambda\left(k_z - \frac{q}{2}\right) - h \end{bmatrix}. \quad (9)$$

By diagonalizing the matrix $[\mathcal{H}_{BdG}]$, we thus obtain the eigenvalue $E_{\mathbf{k}}$ and the vector $[u_{\mathbf{k}\uparrow}, u_{\mathbf{k}\downarrow}, v_{\mathbf{k}\uparrow}, v_{\mathbf{k}\downarrow},]^T$. Actually, we obtain the field operator for Bogoliubov quasiparticles,

$$\alpha_{\mathbf{k}} = \int \left[u_{\mathbf{k}\uparrow}^* e^{-iqz/2}\psi_\uparrow(\mathbf{x}) + u_{\mathbf{k}\downarrow}^* e^{-iqz/2}\psi_\downarrow(\mathbf{x}) + v_{\mathbf{k}\uparrow}^* e^{+iqz/2}\psi_\uparrow^+(\mathbf{x}) + v_{\mathbf{k}\downarrow}^* e^{+iqz/2}\psi_\downarrow^+(\mathbf{x})\right] e^{-i\mathbf{kx}} d\mathbf{x}. \quad (10)$$

Let us now rewrite the mean-field Hamiltonian into the form,

$$\mathcal{H}_{MF} = \frac{1}{2}\sum_{\mathbf{k}} E_{\mathbf{k}}\alpha_{\mathbf{k}}^+\alpha_{\mathbf{k}} - \frac{\Delta^2}{U_0}V + \sum_{\mathbf{k}} \xi_{\mathbf{k}}. \quad (11)$$

Note that, for the Bogoliubov Hamiltonian, we always have the *particle-hole* symmetry, which means that for every solution with $E_{\mathbf{k}} \geq 0$ (say particle, $\alpha_{\mathbf{k}}$), we must have another solution (hole, $\bar{\alpha}_{-\mathbf{k}}$) with $\bar{E}_{-\mathbf{k}} = -E_{\mathbf{k}} \leq 0$. These two solutions are physically the same. Thus, we may rewrite the Hamiltonian,

$$\mathcal{H}_{MF} = \frac{1}{2}\sum_{\mathbf{k},E_{\mathbf{k}}\geq 0}\left(E_{\mathbf{k}}\alpha_{\mathbf{k}}^+\alpha_{\mathbf{k}} - E_{\mathbf{k}}\bar{\alpha}_{-\mathbf{k}}\bar{\alpha}_{-\mathbf{k}}^+\right) - \frac{\Delta^2}{U_0}V + \frac{1}{2}\sum_{\mathbf{k}}\left(\xi_{\mathbf{k}+q/2\mathbf{e}_z} + \xi_{\mathbf{k}-q/2\mathbf{e}_z}\right) \quad (12)$$

$$= \frac{1}{2}\sum_{\mathbf{k},E_{\mathbf{k}}\geq 0} E_{\mathbf{k}}\left(\alpha_{\mathbf{k}}^+\alpha_{\mathbf{k}} + \bar{\alpha}_{-\mathbf{k}}^+\bar{\alpha}_{-\mathbf{k}}\right) - \frac{\Delta^2}{U_0}V + \frac{1}{2}\sum_{\mathbf{k}}\left(\xi_{\mathbf{k}+q/2\mathbf{e}_z} + \xi_{\mathbf{k}-q/2\mathbf{e}_z}\right) - \frac{1}{2}\sum_{\mathbf{k},E_{\mathbf{k}}\geq 0} E_{\mathbf{k}}. \quad (13)$$

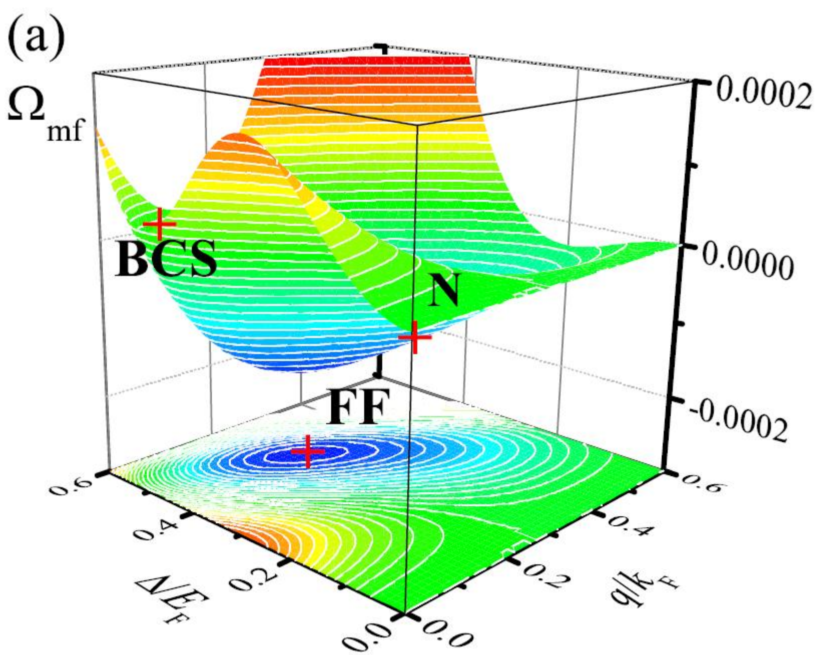
Mean-field theory : general treatment

B. Thermodynamic potential

For given chemical potential μ and temperature T, we have two independent parameters in the BdG equation: the strength of gap parameter Δ and the FF momentum q. These two parameters should be determined by minimizing the grand thermodynamic potential, which takes the following form,

$$\frac{\Omega}{V} = \left[\frac{1}{2V}\sum_{\mathbf{k}}\left(\xi_{\mathbf{k}+q/2\mathbf{e}_z} + \xi_{\mathbf{k}-q/2\mathbf{e}_z}\right) - \frac{1}{2V}\sum_{\mathbf{k},E_{\mathbf{k}}\geq 0} E_{\mathbf{k}}\right] - \frac{\Delta^2}{U_0} - \frac{k_B T}{V}\sum_{E_{\mathbf{k}}\geq 0}\ln\left[1+e^{-\frac{E_{\mathbf{k}}}{k_B T}}\right], \quad (14)$$

where the last term is from the first term in Eq. (13).



$$\begin{cases} \dfrac{\partial\Omega}{\partial\Delta} = 0 \ (gap\ equation) \\ \dfrac{\partial\Omega}{\partial q} = 0 \ (gap\ equation) \\ \dfrac{\partial\Omega}{\partial\mu} = n \ (number\ equation) \end{cases}$$

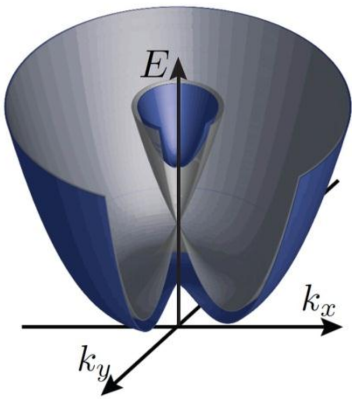
Mean-field theory : general treatment

To calculate the physical quantities of interest, we express the Nambu spinor in terms of the field operators of Bogoliubov quasiparticles.

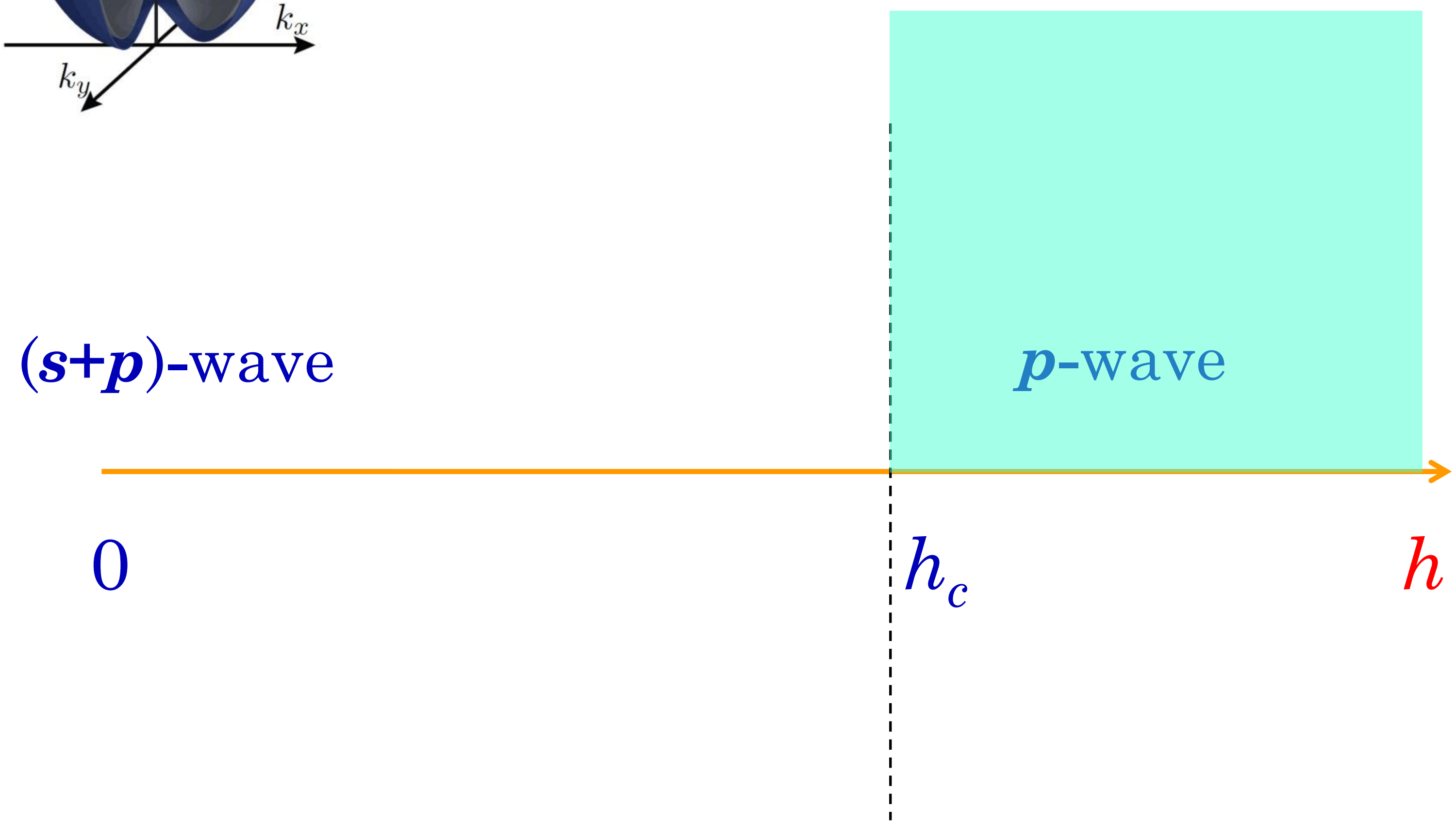
Note that, in the presence of harmonic traps, the mean-field treatment will be a bit different (to be discussed later).

Fluctuations are difficult to handle…

P-wave superfluids?



$V_{SO} = \lambda_R \left(+ k_y \sigma_x - k_x \sigma_y \right)$ **and Zeeman field *h***



Anisotropic superfluidity (no Zeeman field)

Let us focus on Rashba spin-orbit coupling…

Our work:

– PRL **107**, 195304 (2011);
– PRA **84**, 063618 (2011).

Others:

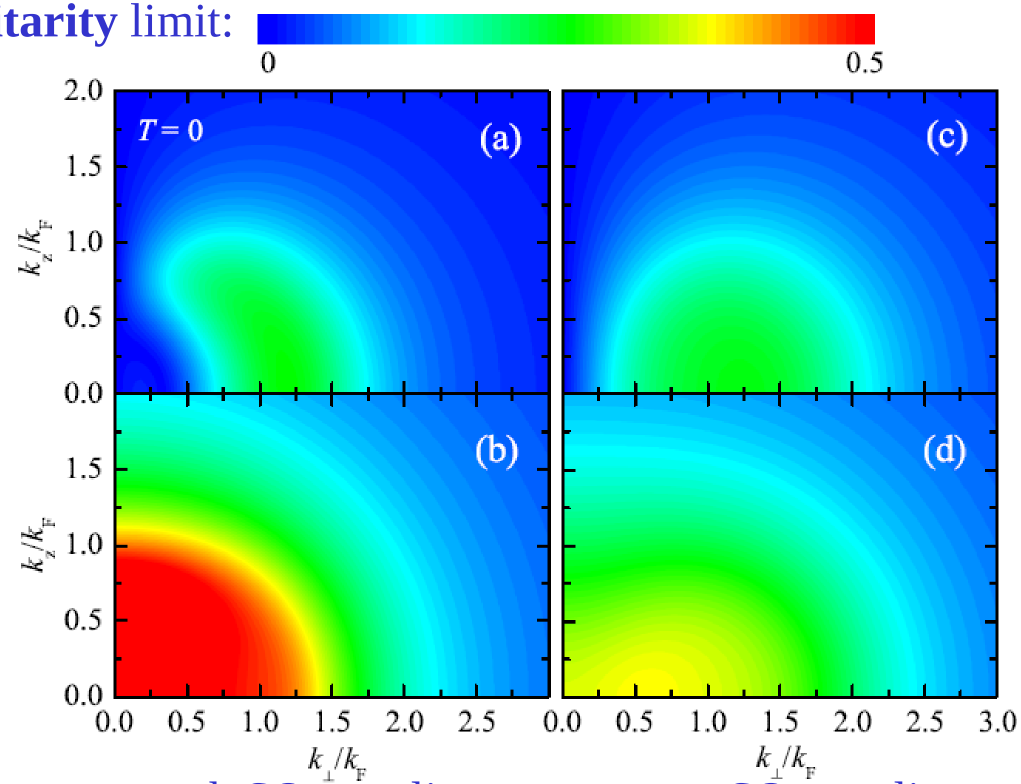
– Shenoy *et al*., PRB (2011);
– Iskin *et al*., PRL (2011);
– Sade Melo *et al*., PRA (2012);
– ……

Anisotropic superfluidity (h=0): Condensed rashbons

For the condensed phase, we solve the mean-field action:

$$S_0 = \int_0^\beta d\tau \int d\mathbf{r} \left(-\frac{\Delta_0^2}{U_0}\right) - \frac{1}{2}\mathrm{Tr}\ln\left[-\mathcal{G}_0^{-1}\right] + \frac{\beta}{V}\sum_{\mathbf{k}} \xi_{\mathbf{k}}$$

In the **unitarity** limit:



weak SO coupling — stronger SO coupling

triplet p-wave pairing

$|\langle \psi_{\boldsymbol{k}\uparrow} \psi_{-\boldsymbol{k}\uparrow} \rangle|$

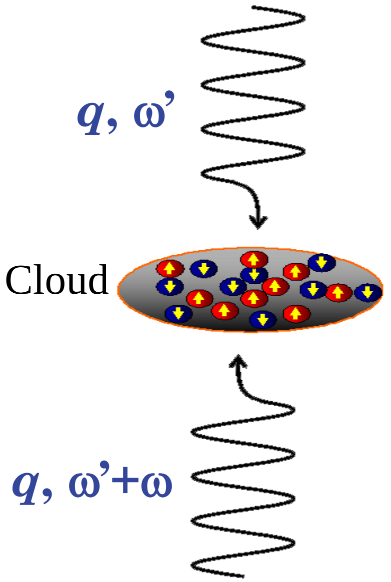
singlet s-wave pairing

$|\langle \psi_{\boldsymbol{k}\uparrow} \psi_{-\boldsymbol{k}\downarrow} \rangle|$

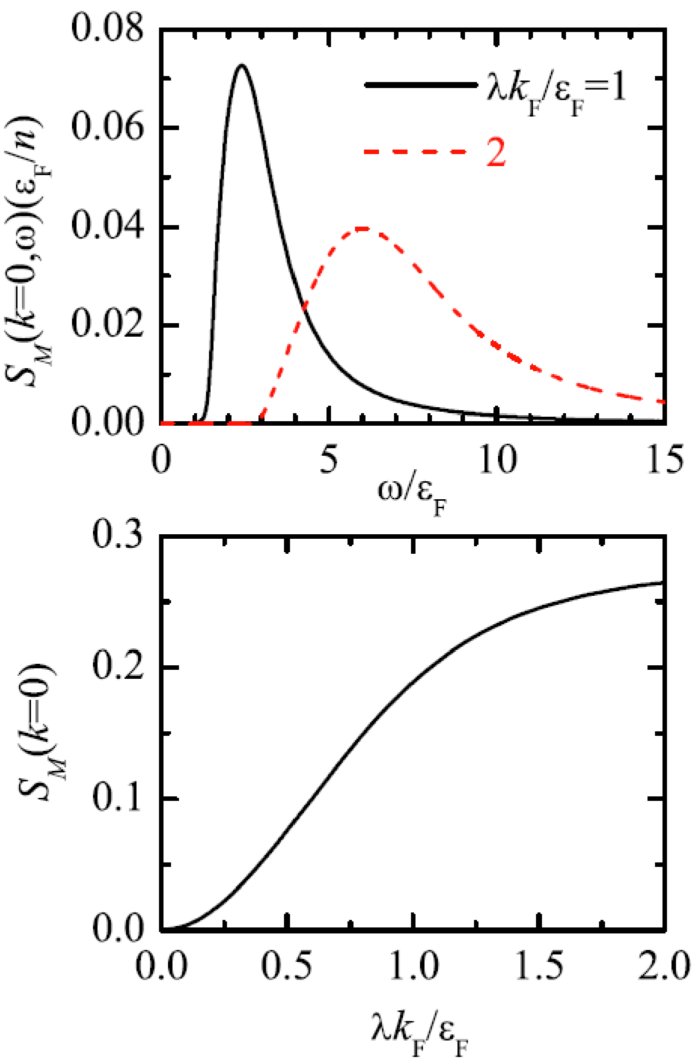
Rashbons condense into a mixed singlet-triplet state!

See also, Gor'kov & Rashba, *Phys. Rev. Lett.* 2001.

Anisotropic superfluidity: Condensed Rashbons

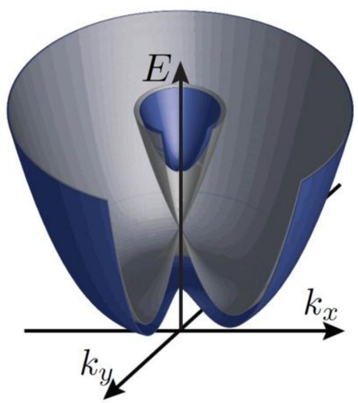


Swinburne:
Bragg spectroscopy

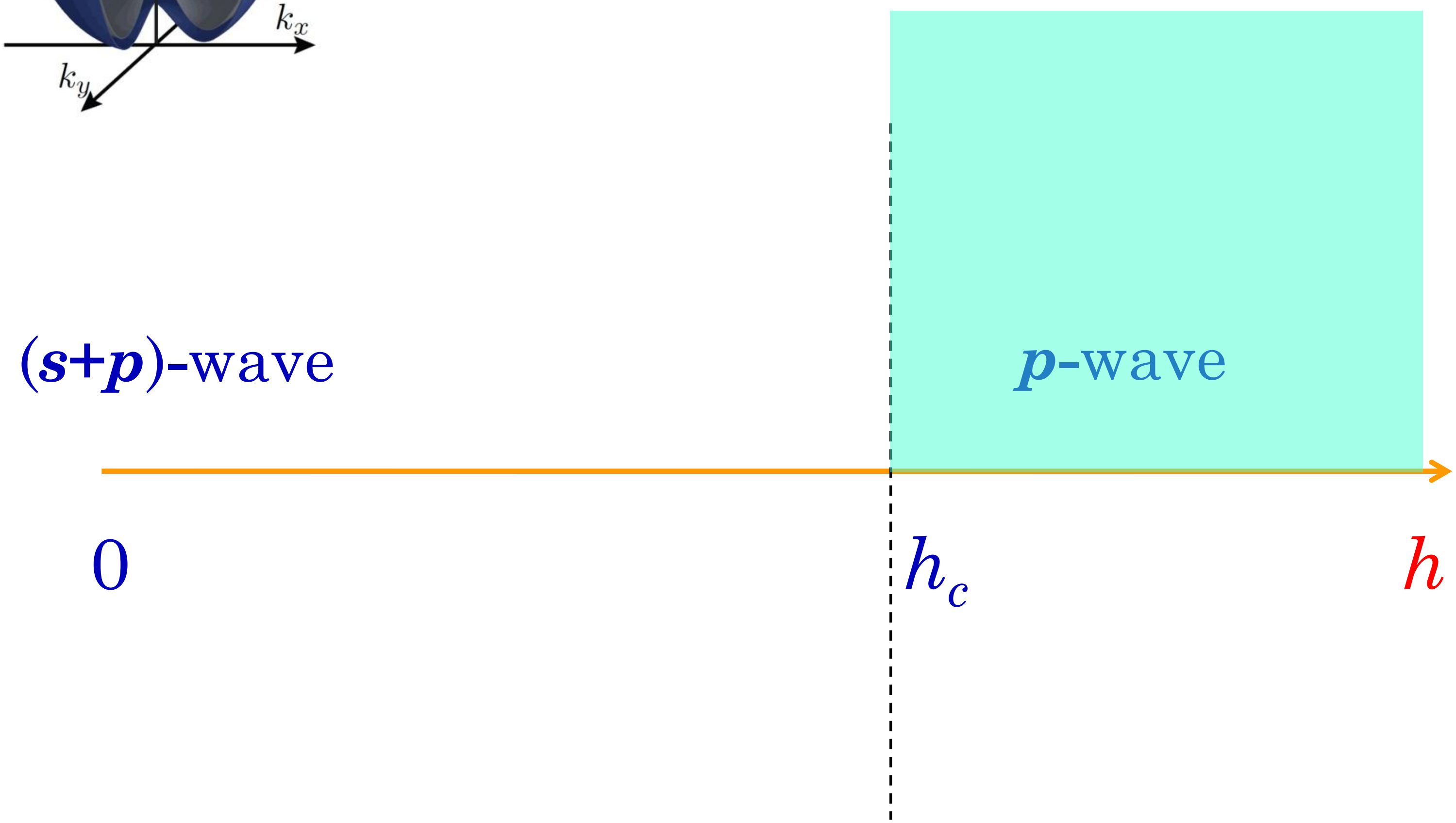


The smoking-gun of anisotropic superfluidity:
spin dynamic structure factor at long wavelength

P-wave superfluids?



$$V_{\mathrm{SO}} = \lambda_{\mathrm{R}} \left(+ k_y \sigma_x - k_x \sigma_y \right)$$ **and Zeeman field *h***



Topological superfluidity
(out-of-plane Zeeman field)

Our work:
- PRA **85**, 021603(R) (2012);
- PRA **85**, 033622 (2012);
- PRL **110**, 020401 (2013);
- PRA **87**, 013622 (2013).

Others:
- Mueller *et al*., PRA (2012);
- Sade Melo *et al*., PRL (2012);
- ……

P-wave superconductors

2D chiral *p*-wave superconductor: $\boxed{\Delta(\mathbf{k}) = \Delta_0\left(k_x + ik_y\right)}$

$$H = \sum_{\mathbf{k}}\left(\frac{\mathbf{k}^2}{2m} - \mu\right)c_{\mathbf{k}}^{\dagger}c_{\mathbf{k}} + \left(\Delta(\mathbf{k})c_{\mathbf{k}}c_{-\mathbf{k}} + c.c.\right)$$ Read & Green, *PRB* 2000

Defining a **Nambu spinor** $\psi_{\mathbf{k}} = \left(c_{\mathbf{k}}, c_{-\mathbf{k}}^{+}\right)^T$

$$\boxed{H = \frac{1}{2}\sum_{\mathbf{k}}\begin{pmatrix} c_{\mathbf{k}}^{+} & c_{-\mathbf{k}} \end{pmatrix}\begin{bmatrix} \frac{\mathbf{k}^2}{2m} - \mu & -\Delta^*(\mathbf{k}) \\ -\Delta(\mathbf{k}) & -\frac{\mathbf{k}^2}{2m} + \mu \end{bmatrix}\begin{pmatrix} c_{\mathbf{k}} \\ c_{-\mathbf{k}}^{+} \end{pmatrix}}$$

$$H = \frac{1}{2}\sum_{\mathbf{k}}\psi_{\mathbf{k}}^{+}\left[\left(\frac{k^2}{2m} - \mu\right)\sigma_z - \Delta_0\left(k_x\sigma_x + k_y\sigma_y\right)\right]\psi_{\mathbf{k}}$$

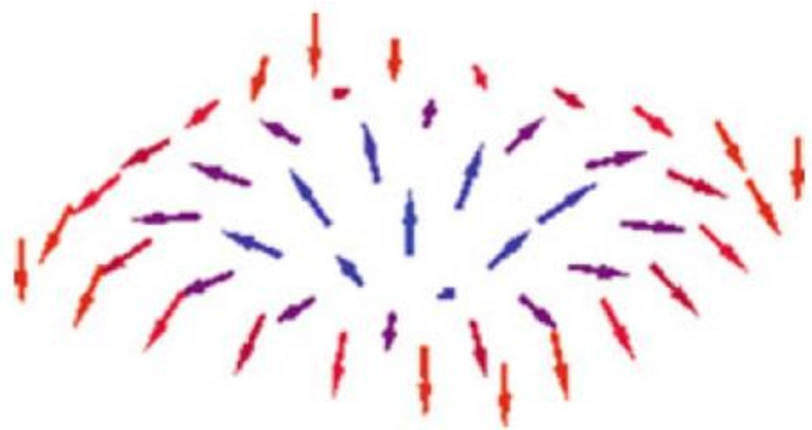
P-wave superconductors

$$H = \frac{1}{2}\sum_{\mathbf{k}} \psi_{\mathbf{k}}^{+}\left[\left(\frac{k^2}{2m} - \mu\right)\sigma_z - \Delta_0\left(k_x\sigma_x + k_y\sigma_y\right)\right]\psi_{\mathbf{k}}$$

Consider the spin vector in *k* space: $-\boldsymbol{B}(\boldsymbol{k})\cdot\boldsymbol{\sigma}$ where $B_z(k) = \mu - \dfrac{k^2}{2m}$

$B_x(k) = \Delta_0 k_x$

$B_y(k) = \Delta_0 k_y$

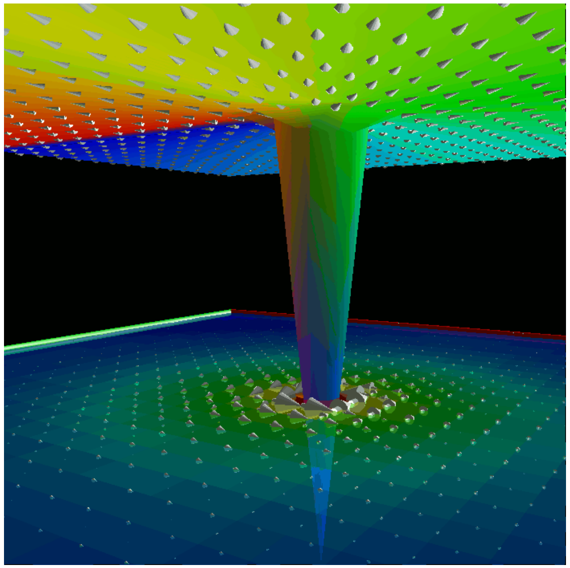


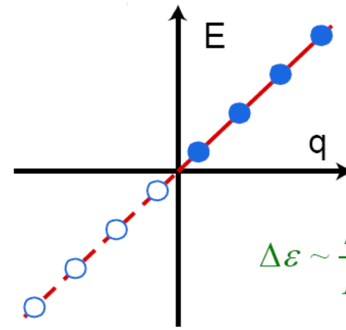
! applicable to 1D as well:

A topological defect – **Skyrmion** – forms when **μ > 0**.

P-wave superconductors: zero vortex-core mode

consider vortex state…





CdGM bound states

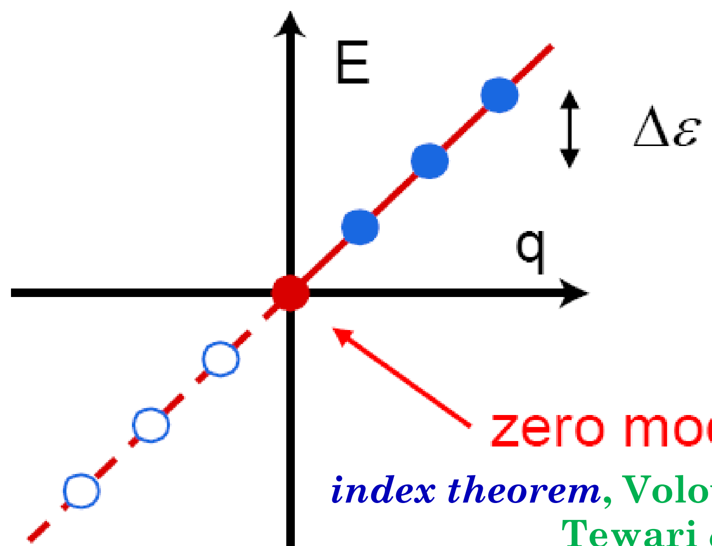
$\Delta\varepsilon \sim \frac{\Delta^2}{E_F}$ $\xi \sim \hbar v_f/\Delta$

Caroli, de Gennes, Matricon theory ('64)

if conventional superconductors

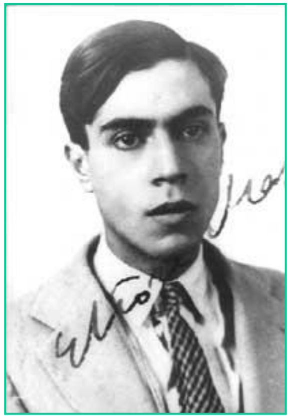
if weak *p*-wave superconductors:

Kopnin and Salomaa, 91



***index theorem*, Volovik JETP 93**
Tewari *et al*, PRL 07

Majorana fermions



Simple idea of Majorana (1937):

An ordinary Dirac fermion = two real fermions

$$c = \gamma_1 - i\gamma_2$$

Majorana fermion: particle is its own antiparticle

$$\boxed{\gamma = \gamma^+}$$

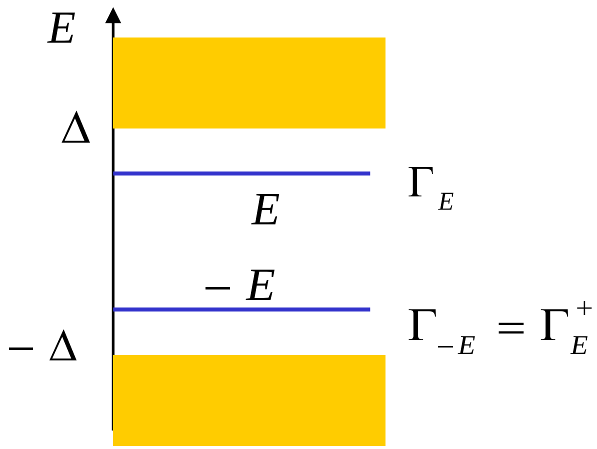
Majorana fermions

Particle-hole symmetry in BdG equations:

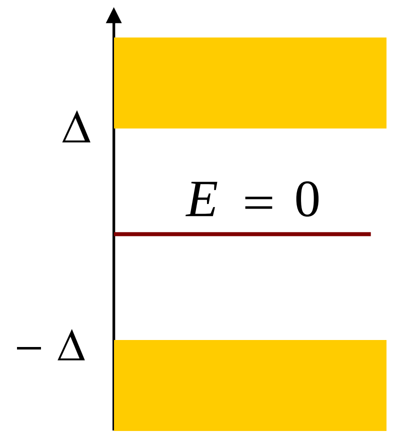
$$u_\sigma(\mathbf{r}) \to v_\sigma^*(\mathbf{r})$$

$$E_\eta \to -E_\eta$$

"non-topological"



"topological"



$$\Gamma_{E=0} = \Gamma_{E=0}^{+} = \gamma$$

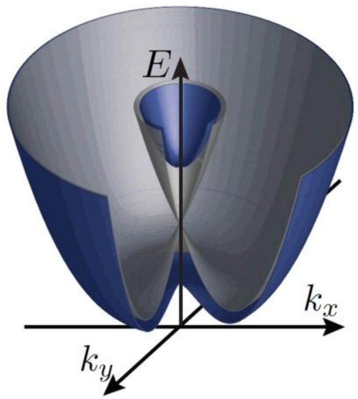
Majorana fermions !!!

$$1.\quad u_\sigma = +v_\sigma^*$$

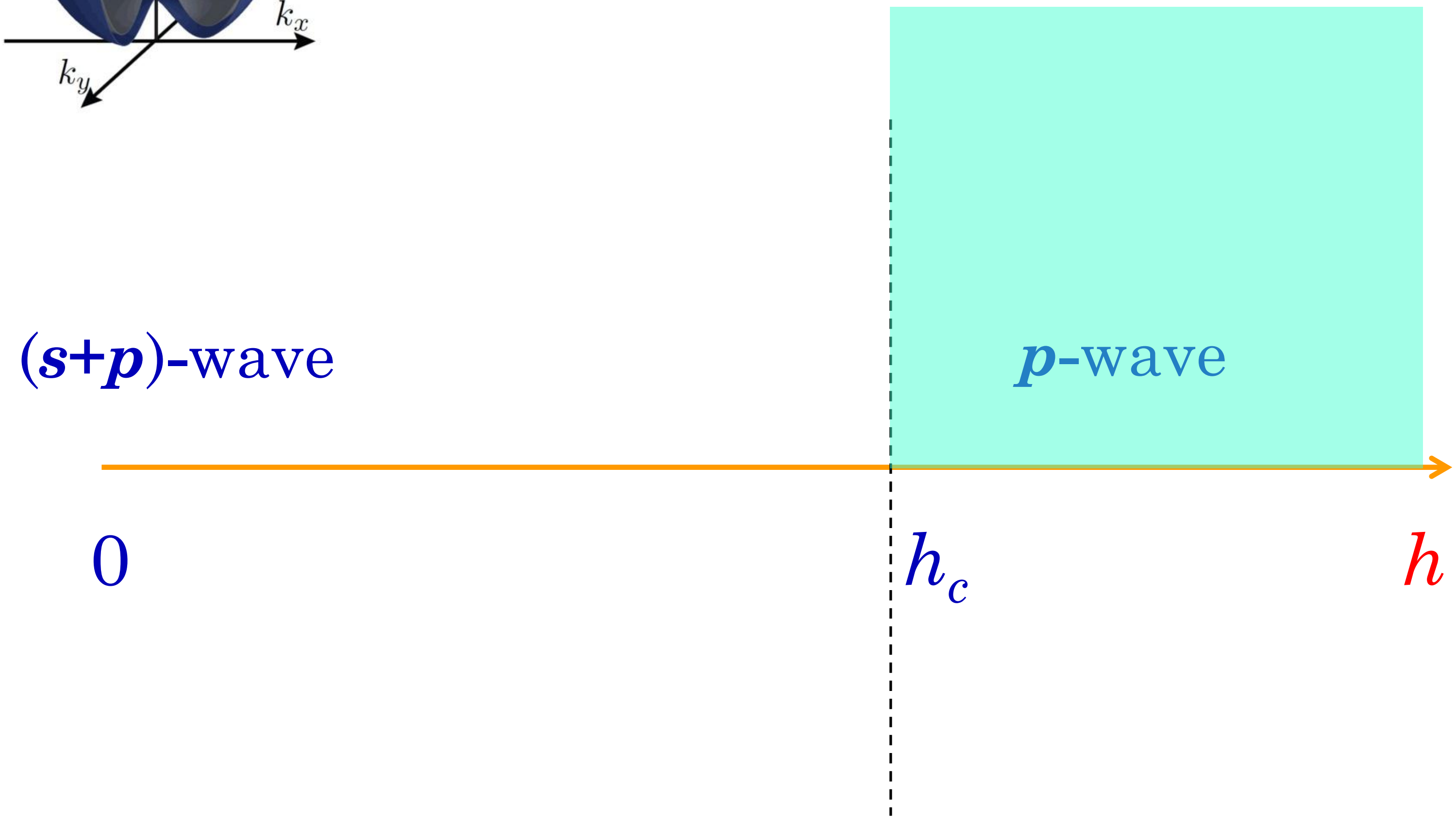
$$2.\quad u_\sigma = -v_\sigma^*$$

transition occurs only if energy gap close.

P-wave superfluids?



$$V_{\mathrm{SO}} = \lambda_{\mathrm{R}}\left(+k_y\sigma_x - k_x\sigma_y\right)$$ **and Zeeman field *h***

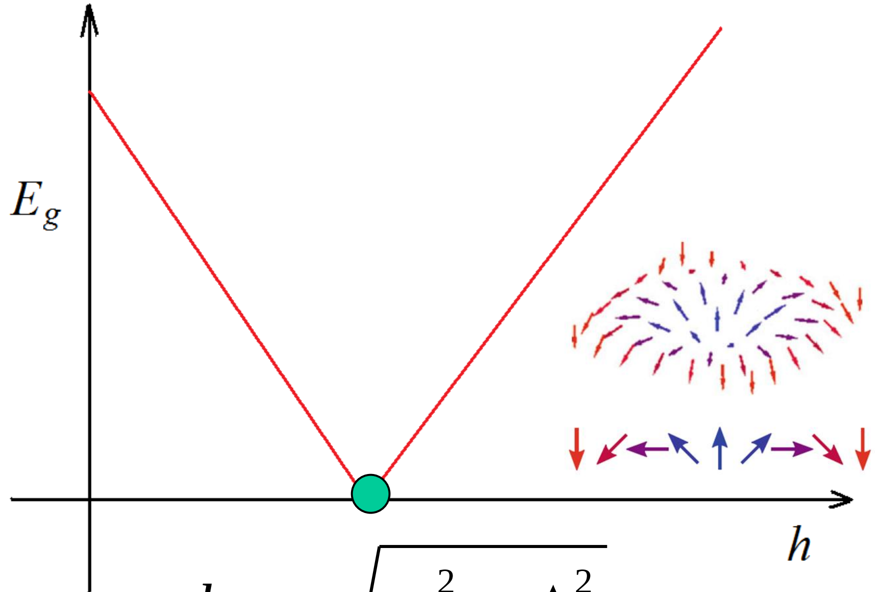


Recipe for topological superfluid: (C. Zhang, *PRL* 08 for cold-atoms)

Feshbach *s*-wave resonance ☺

Rashba spin-orbit coupling ☺

Large Zeeman field ☺



$$h_c = \sqrt{\mu^2 + \Delta^2}$$

2D trapped SOC atomic Fermi gases + BdG

Hamiltonian

$$\boxed{\mathcal{H} = \int d\mathbf{r}[\mathcal{H}_0(\mathbf{r}) + \mathcal{H}_I(\mathbf{r})]}$$

Single-particle Hamiltonian (Rashba SOC)

$$\mathcal{H}_0(\mathbf{r}) = \sum_{\sigma=\uparrow,\downarrow} \psi_\sigma^\dagger \mathcal{H}_\sigma^S(\mathbf{r})\psi_\sigma + \left[\psi_\uparrow^\dagger V_{SO}(\mathbf{r})\psi_\downarrow + \text{H.c.}\right]$$

$$V_{SO}(\mathbf{r}) = -i\lambda(\partial_y + i\partial_x)$$

$$\mathcal{H}_\sigma^S = -\hbar^2\nabla^2/(2M) + M\omega_\perp^2 r^2/2 - \mu - h\sigma_z$$

Interaction Hamiltonian

$$\mathcal{H}_I(\mathbf{r}) = U_0\psi_\uparrow^\dagger(\mathbf{r})\psi_\downarrow^\dagger(\mathbf{r})\psi_\downarrow(\mathbf{r})\psi_\uparrow(\mathbf{r})$$

Mean-field BdG theory:

$$\boxed{\mathcal{H}_{BdG}\Psi_\eta(\mathbf{r}) = E_\eta\Psi_\eta(\mathbf{r})}$$

$$\boxed{\Psi_\eta(\mathbf{r}) = [u_{\uparrow\eta}, u_{\downarrow\eta}, v_{\uparrow\eta}, v_{\downarrow\eta}]^T}$$

$$\mathcal{H}_{BdG} = \begin{bmatrix} \mathcal{H}_\uparrow^S(\mathbf{r}) & V_{SO}(\mathbf{r}) & 0 & -\Delta(\mathbf{r}) \\ V_{SO}^\dagger(\mathbf{r}) & \mathcal{H}_\downarrow^S(\mathbf{r}) & \Delta(\mathbf{r}) & 0 \\ 0 & \Delta^*(\mathbf{r}) & -\mathcal{H}_\uparrow^S(\mathbf{r}) & V_{SO}^\dagger(\mathbf{r}) \\ -\Delta^*(\mathbf{r}) & 0 & V_{SO}(\mathbf{r}) & -\mathcal{H}_\downarrow^S(\mathbf{r}) \end{bmatrix}$$

Self-consistency:

$$\Delta = -(U_0/2)\sum_\eta[u_{\uparrow\eta}v_{\downarrow\eta}^* f(E_\eta) + u_{\downarrow\eta}v_{\uparrow\eta}^* f(-E_\eta)]$$

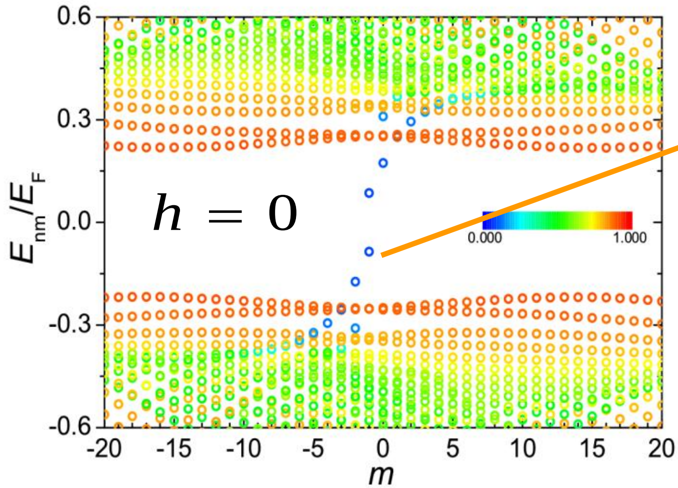
$$n_\sigma(\mathbf{r}) = (1/2)\sum_\eta[|u_{\sigma\eta}|^2 f(E_\eta) + |v_{\sigma\eta}|^2 f(-E_\eta)]$$

Single vortex

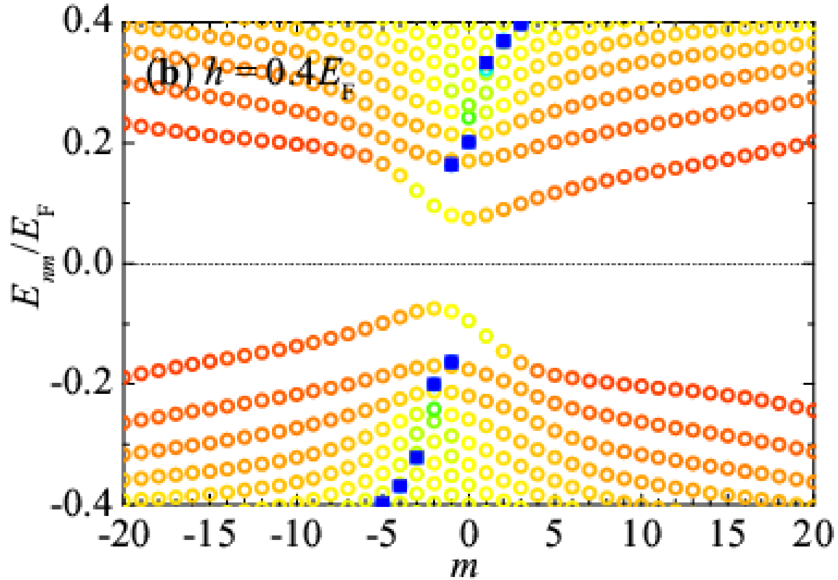
$$\boxed{\Delta(\mathbf{r}) = \Delta(r)e^{-i\varphi}}$$

Low-lying Bogoliubov quasi-particles

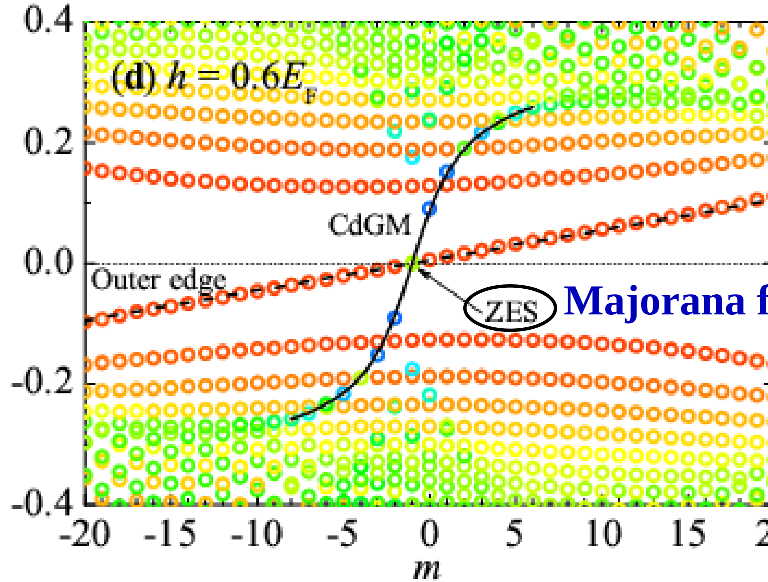
Quasi-particle excitation spectrum in the presence of a **single** vortex $\lambda k_F / E_F = 1, \quad T = 0, \quad E_a = 0.2 E_F$



Non-topological

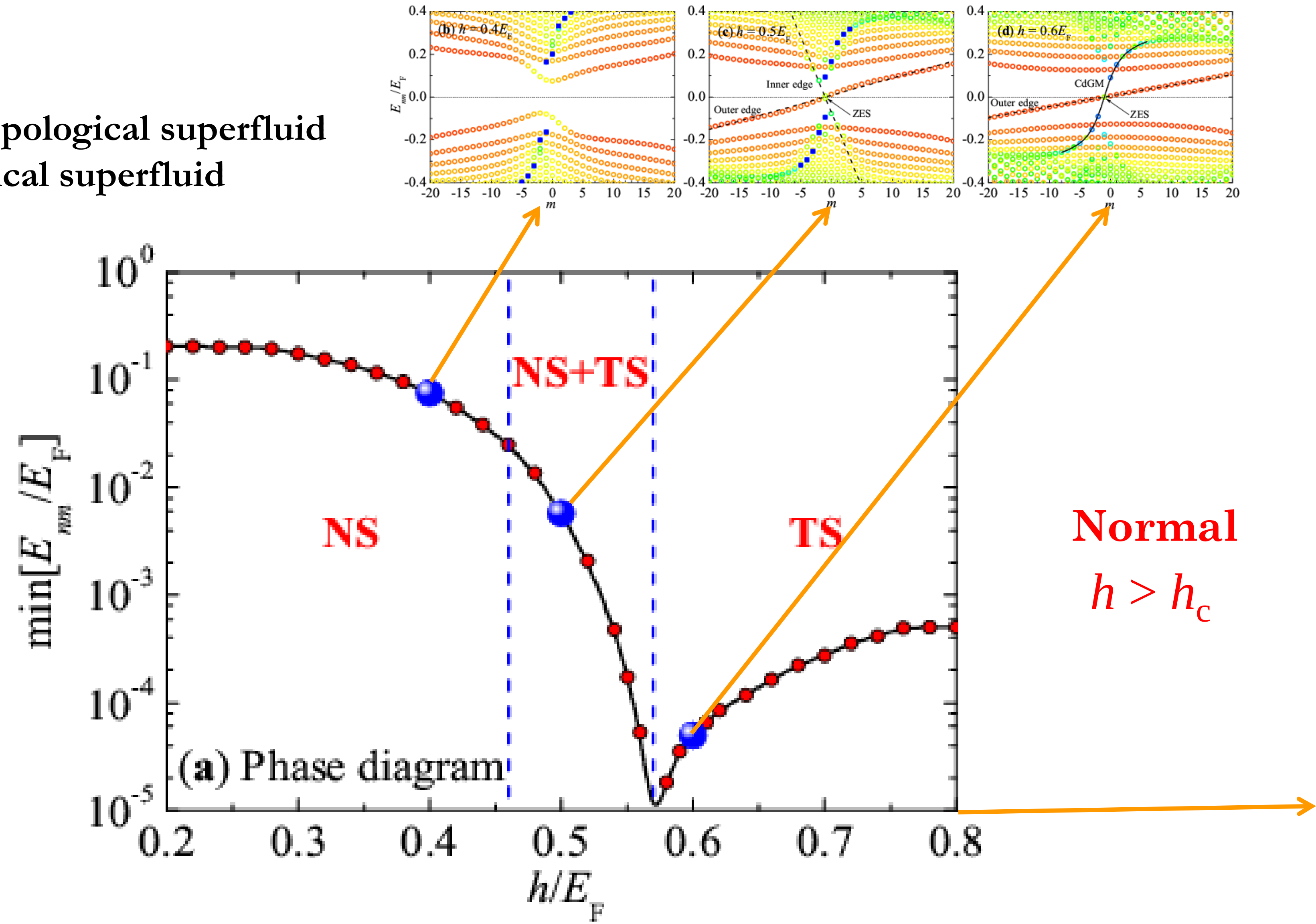


Topological



Phase diagram

NS: non-topological superfluid
TS: topolgical superfluid

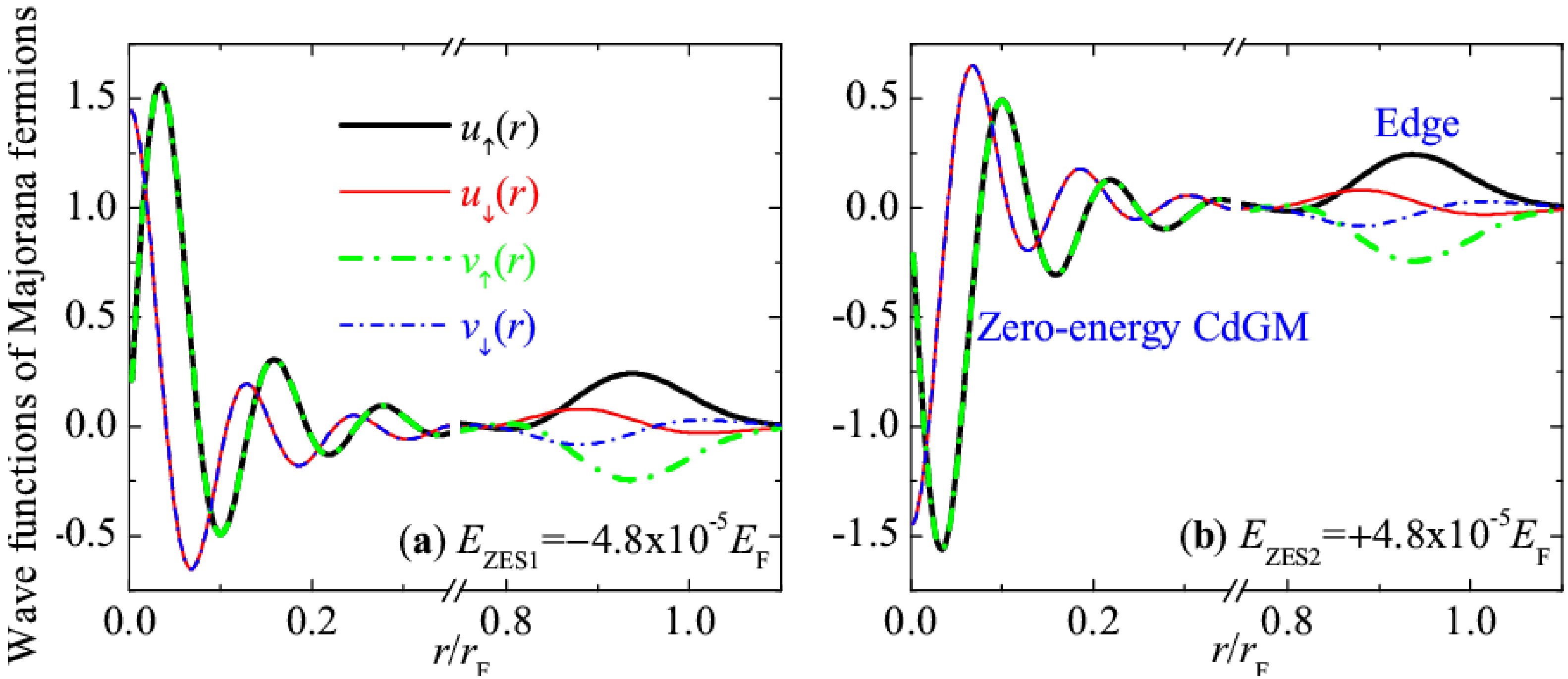


Normal
$h > h_c$

X.-J. Liu, L. Jiang, H. Pu, and HH, *Phys. Rev. A* **85**, 021603(R) (2012).

Wavefunctions of Majorana fermions

$h = 0.6E_F$ **(Topological superfluid phase)**

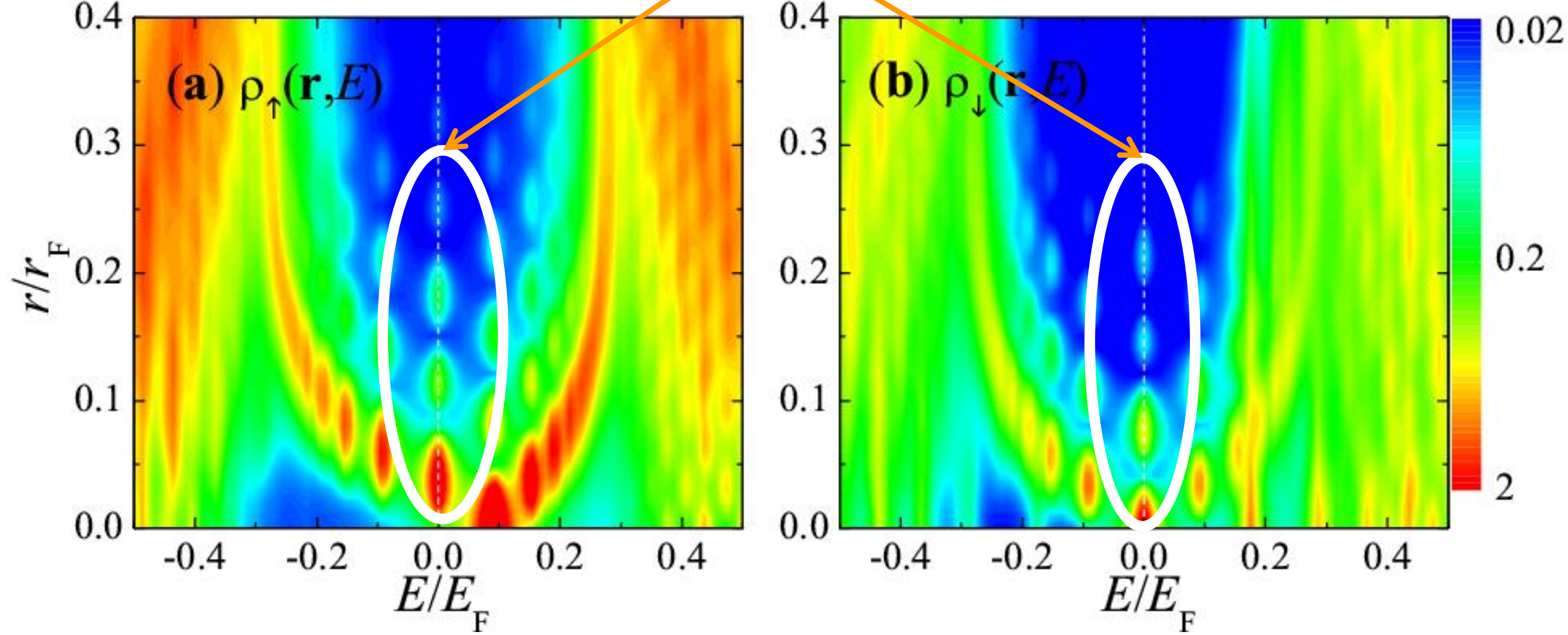


1. **Bond and anti-bond hybridization** $u_\sigma = v_\sigma^*$ **and** $u_\sigma = -v_\sigma^*$ **solutions.**
2. **Quasiparticle tunneling** $\longrightarrow$ **energy splitting.**

Probing Majorana fermions in 2D

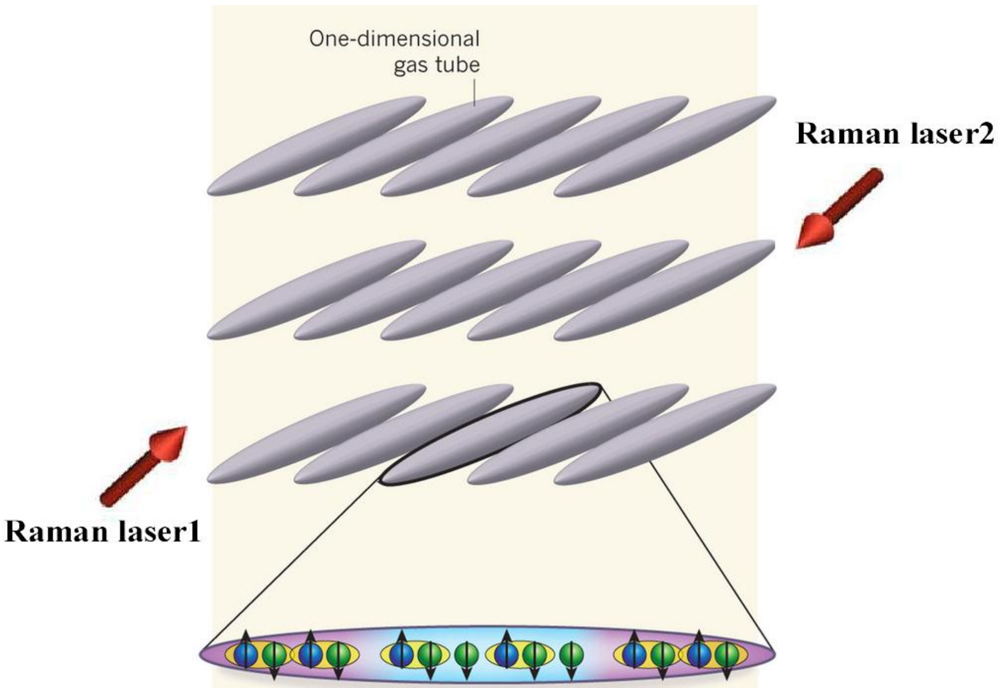
Local density of states: $$\rho_\sigma(r,E)=\frac{1}{2}\sum_\eta\left[|u_{\sigma\eta}|^2\,\delta(E-E_\eta)+|v_{\sigma\eta}|^2\,\delta(E+E_\eta)\right]$$

$$\boxed{\rho_\sigma(r,0)\propto|u_{\sigma\eta}(r)|^2=|v_{\sigma\eta}(r)|^2}$$



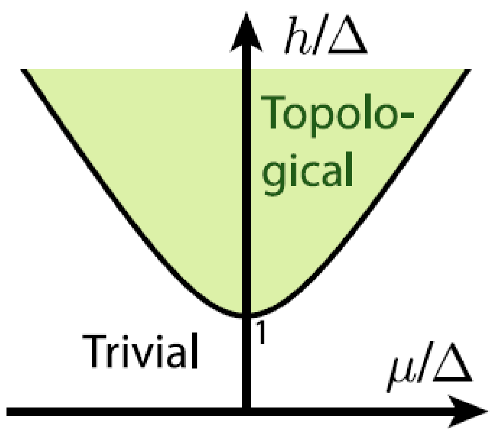
Directly: Use the spatially resolved rf-spectroscopy (cold-atom STM).

Equal Rashba and Dresselhaus SOC



$$\mathcal{H}_0 = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V_T(x) - \mu - h\sigma_z + \lambda \hat{k}_x \sigma_y$$

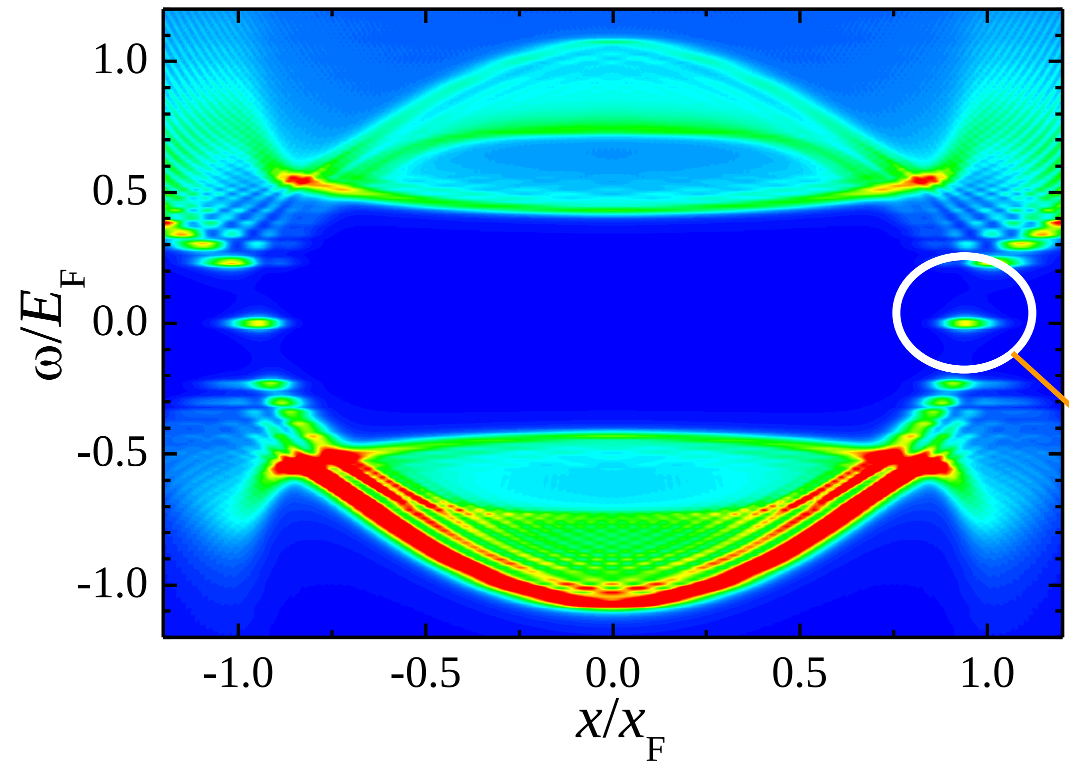
$$h = \frac{\Omega}{2}$$



$h > \sqrt{\Delta^2 + \mu^2}$ (topological criterion)

Characterizing 1D topological superfluid (rf)

Majorana fermions by spatially-resolved rf-spectroscopy (cold-atom STM):



Majorana fermions

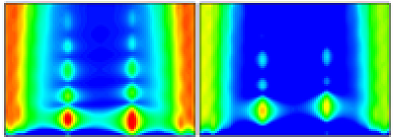
Requires: $T < 0.1T_F$

Currently for ^{40}K atoms: $T = 0.6T_F$

X.-J. Liu and HH, *Phys. Rev. A* **85**, 033622 (2012).
X.-J. Liu, *Phys. Rev. A* **87**, 013622 (2013).

Physics

Synopsis: Useful Impurities



H. Hu *et al.*, Phys. Rev. Lett. (2013)

Universal Impurity-Induced Bound State in Topological Superfluids

Hui Hu, Lei Jiang, Han Pu, Yan Chen, and Xia-Ji Liu
Phys. Rev. Lett. **110**, 020401 (2013)
Published January 10, 2013

Impurities are not always just unwanted defects that degrade the properties of a solid. Sometimes they can be used as sensitive probes of the physical properties of their host; for instance, magnetic impurities in unconventional superconductors have helped decipher the underlying pairing mechanism. Writing in *Physical Review Letters*, Hui Hu, at the Centre for Atomic Optics and Ultrafast Spectroscopy in Australia, and co-workers discuss how to use magnetic and nonmagnetic impurities to characterize a state of matter that is hard to observe experimentally: a topological superfluid.

Topological superfluids are completely frictionless fluids of fermions in which quantum states are topologically protected from scattering and decoherence. According to theory, topological superfluids would host excitations known as Majorana quasiparticles, which are capable of forming robust quantum superposition states and are therefore of great interest for quantum computing applications. Certain superconductors, nanowires, and three-dimensional topological insulators are conjectured to host a topological superfluid, but to date it hasn't been possible to convincingly confirm the existence of this state of matter in any of these systems.

The authors calculated the electronic states close to an impurity embedded in a topological superfluid. Their findings suggest that an electronic state, bound to the impurity, would emerge in an otherwise gapped spectrum—regardless of the type of impurity or superfluid. Such a midgap state would lead to spectroscopic observables that could provide unambiguous signatures of the topological superfluid state. Further, their calculations show that the wave function of such an impurity state has the same spatial symmetry as a Majorana state, which may suggest the use of controlled impurities in topological superfluids as elementary quantum bits.

Fulde-Ferrell inhomogeneous superfluidity
(in-plane Zeeman field)

Our work:

- PRA **87**, 043613 (R) (2013);
- PRA **88**, 023622 (2013);
- PRA **88**, 043607 (2013);
- NJP **15**, 093037 (2013).

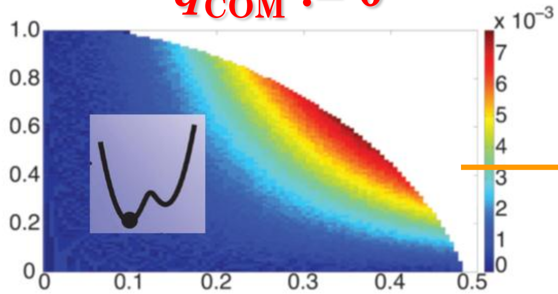
Others:

- C. Zhang *et al*., PRA (2013);
- Yi and Zhang *et al*., PRL (2013);
- Shenoy, PRA (2013);
- Pu *et al*., NJP (2013);
- Zhou *et al*., PRA (2013).

Fulde-Ferrell pairing – a 50-year-old puzzle

$$H = \frac{\hbar^2 \hat{k}^2}{2m} + \lambda_{SO} k_x \sigma_y + \frac{\delta}{2}\sigma_y + \frac{\Omega}{2}\sigma_z$$

q_{COM} **!= 0**



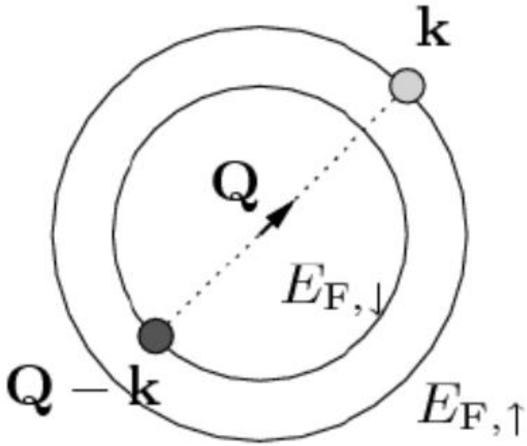
two-particle spectrum

Han Pu: **FF superfluid in the many-body setting?**

Fulde Ferrell

Fulde-Ferrell pairing – a 50-year-old puzzle

- BCS Cooper pairs have zero momentum
- Population imbalance leads to finite-momentum pairs (FF 1964, see also LO)
- Fulde-Ferrell-Larkin-Ovichinnikov (FFLO) instability results in textured states
- Spontaneously breaks translational symmetry



$$Q \propto E_{F\uparrow} - E_{F\downarrow}$$

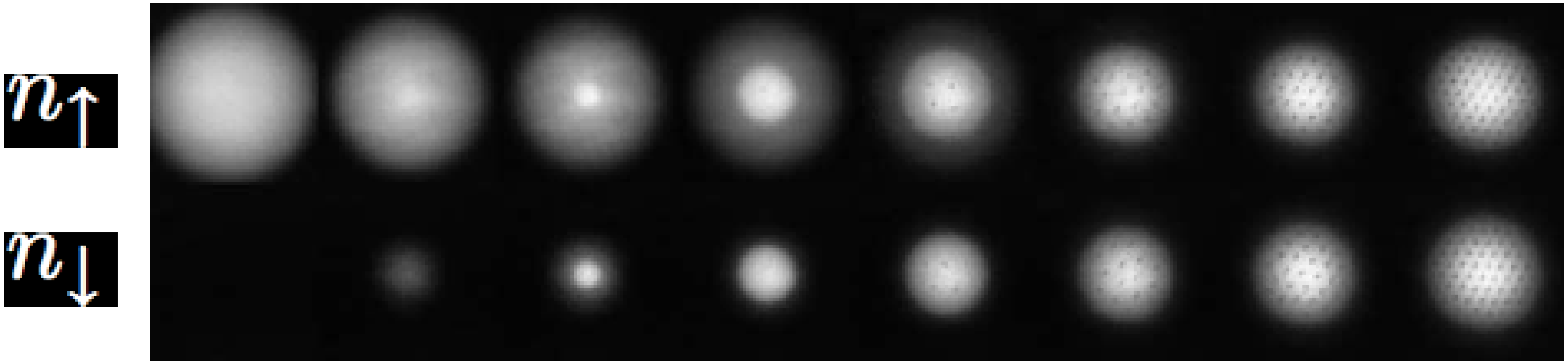
$$\Delta(\mathbf{x}) \propto e^{iQ\cdot\mathbf{x}} \longrightarrow \Delta(\mathbf{x}) \propto \cos(Q\cdot\mathbf{x})$$

FF superfluid   **LO** superfluid

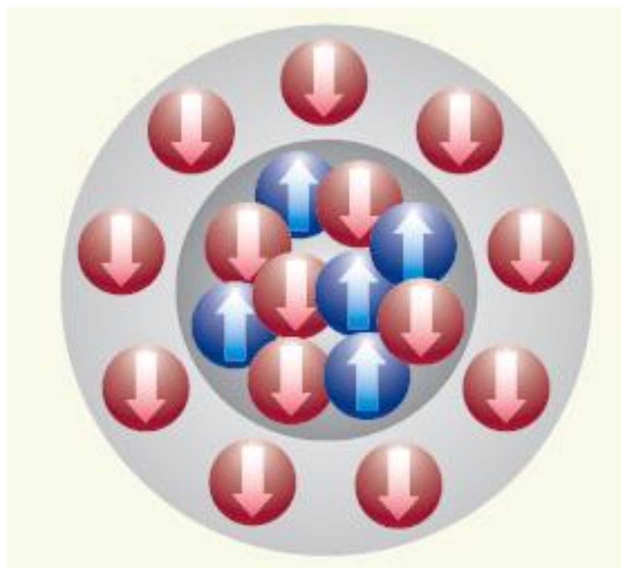
SWINBURNE CENTRE FOR ATOM OPTICS AND ULTRAFAST SPECTROSCOPY

Fulde-Ferrell pairing – a 50-year-old puzzle

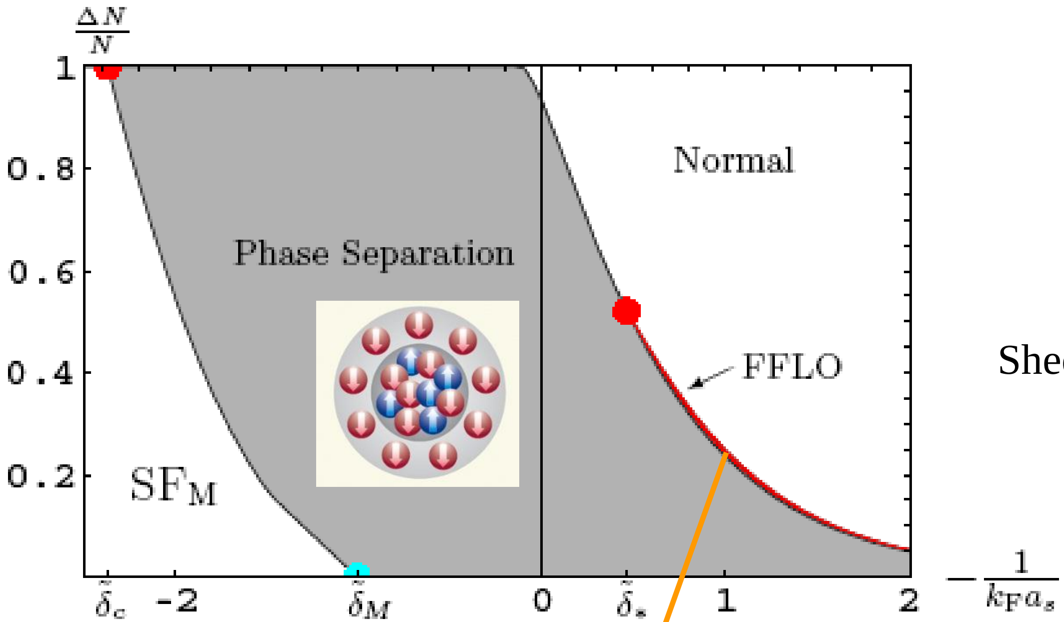
AUSTRALIAN RESEARCH COUNCIL CENTRE OF EXCELLENCE FOR QUANTUM-ATOM OPTICS



M. W. Zwierlein, A. Schirotzek, C. H. Schunck, and W. Ketterle, Science **311**, 492 (2006)



3D trapped Fermi gas: superfluid core with polarized halo…



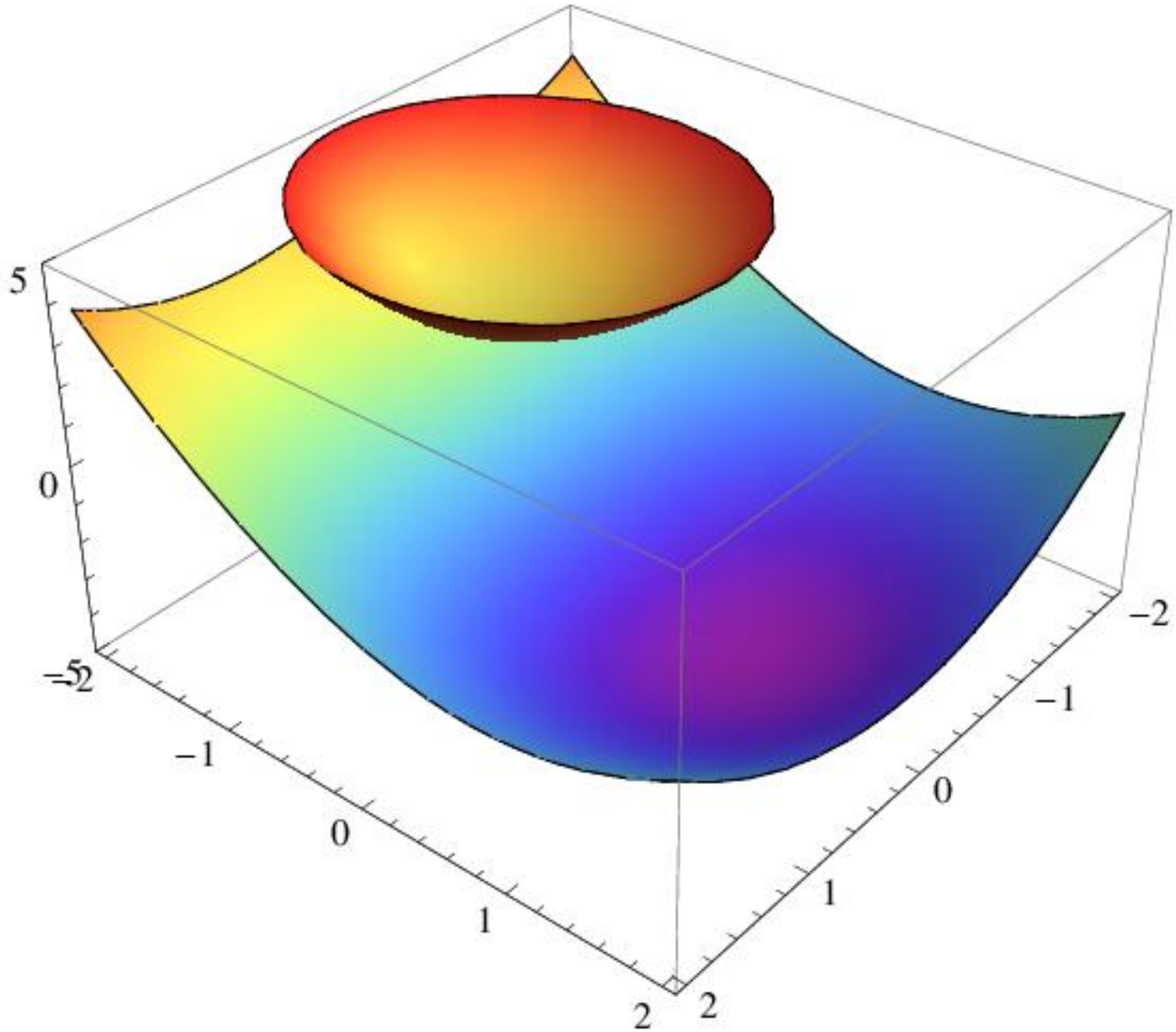
FF(LO) is not favored in 3D.

Sheehy and Radzihovsky, Ann. Phys. (2007)

Enhanced by spin-orbit coupling?

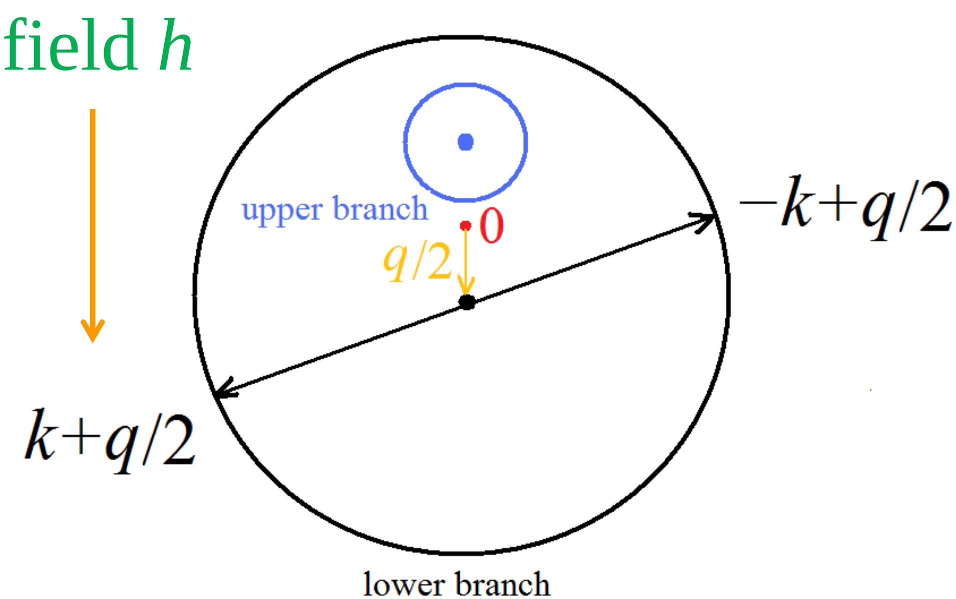
Yes! The deformation of Fermi surfaces due to spin-orbit coupling and in-plane Zeeman field provides another mechanics for FF pairing instability (Barzykin & Gor'kov PRL 2002; now realized by a number of researchers: Han Pu, V. B. Shenoy, C. Zhang, W. Yi, W. Zhang…)

Fermi surfaces (SOC & in-plane field)



Fulde-Ferrell pairing – a 50-year-old puzzle

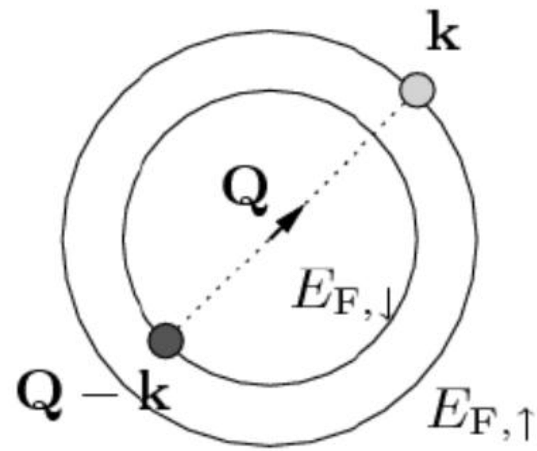
Fermi surfaces (SOC & in-plane field)



$$q \propto h$$

FF superfluid

Fermi surfaces (population imbalance)



$$Q \propto E_{F\uparrow} - E_{F\downarrow}$$

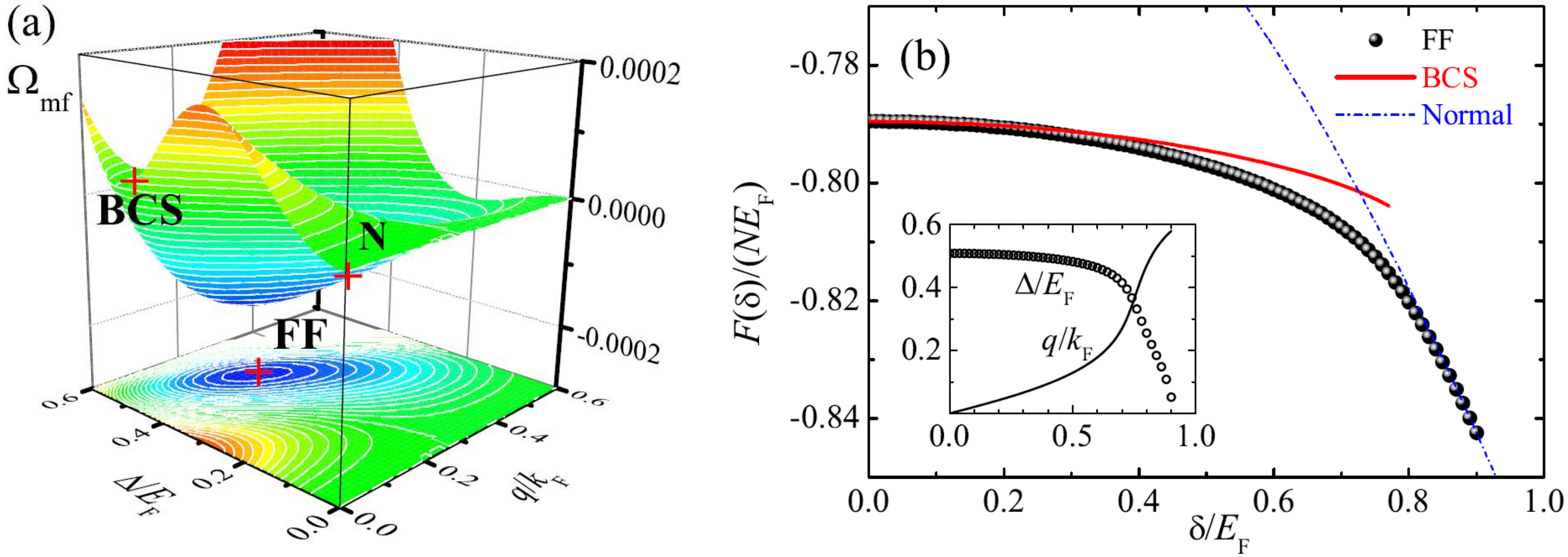
LO superfluid

Fulde-Ferrell pairing instability: ERD-SOC

For the FF superfluid, we minimize the mean-field action by assuming $\Delta_0(r) = \Delta e^{iqr}$

$$S_0 = \int_0^\beta d\tau \int d\mathbf{r} \left(-\frac{\Delta_0^2}{U_0}\right) - \frac{1}{2}\mathrm{Tr}\ln\left[-\mathcal{G}_0^{-1}\right] + \frac{\beta}{V}\sum_{\mathbf{k}} \xi_{\mathbf{k}}$$

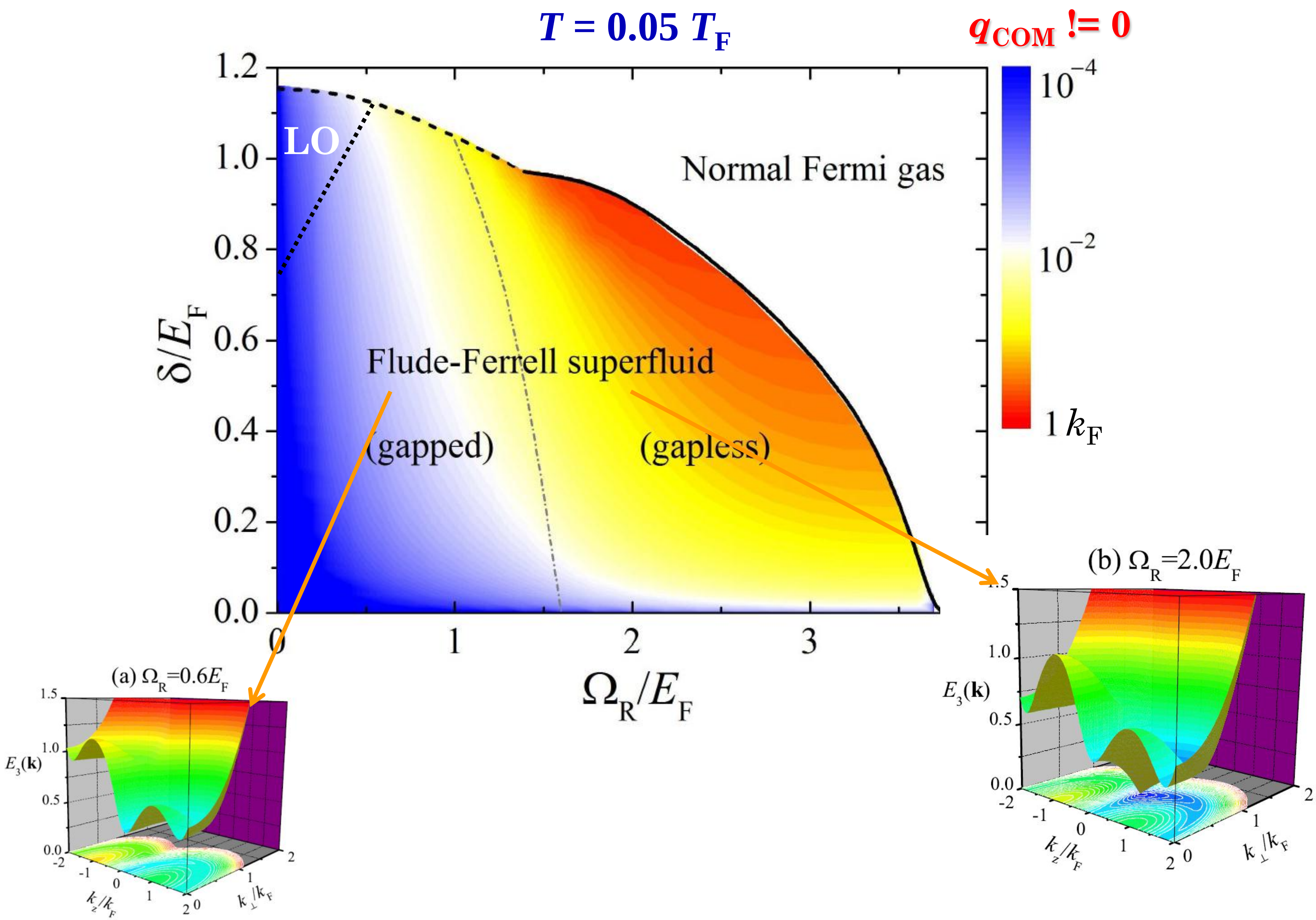
$$\Omega = 2E_{\mathrm{R}}$$



FF is always favorable!

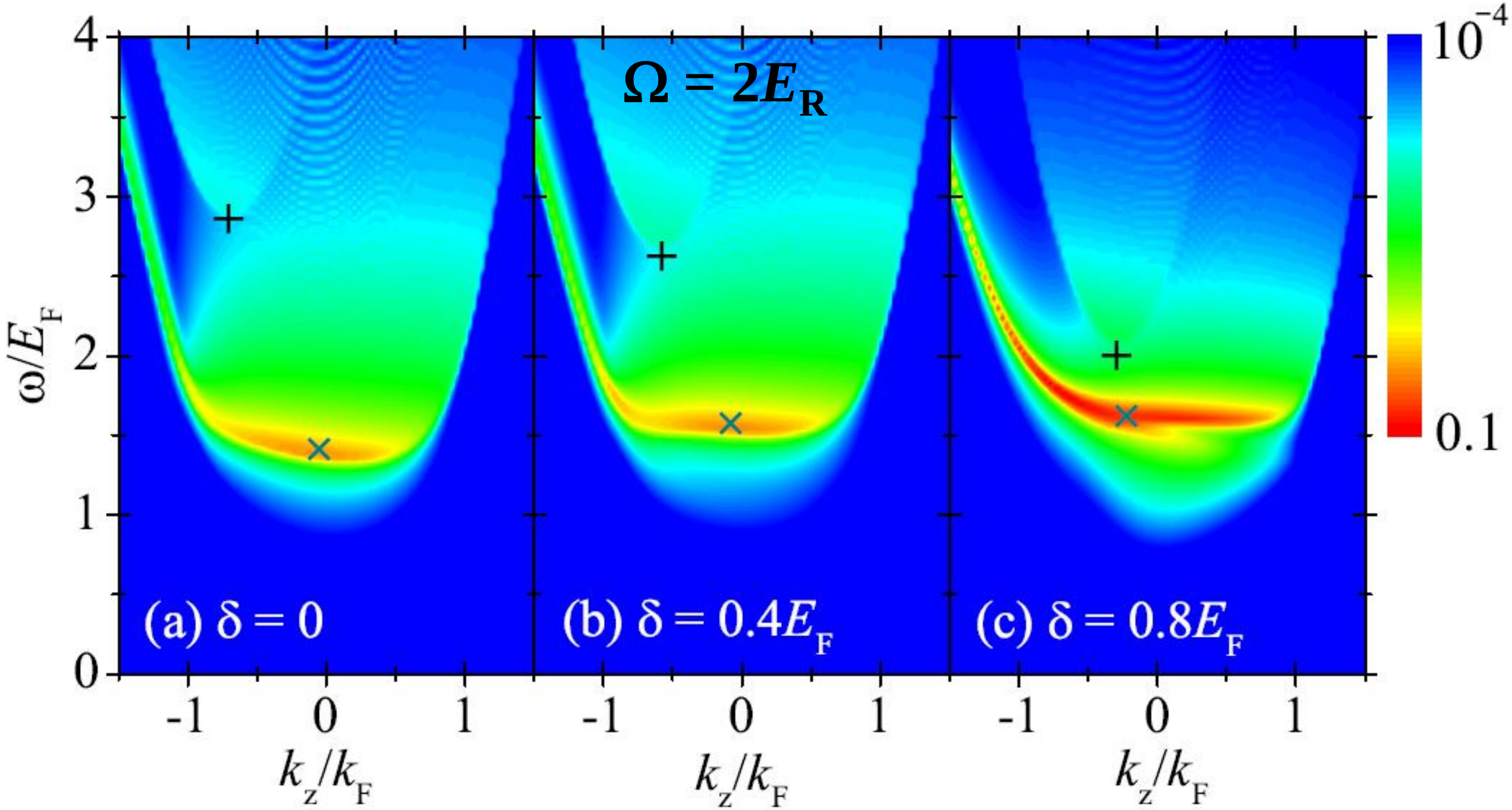
X.-J. Liu and HH, *Phys. Rev. A* **87**, 043616(R) (2013).

Fulde-Ferrell phase diagram: ERDSOC



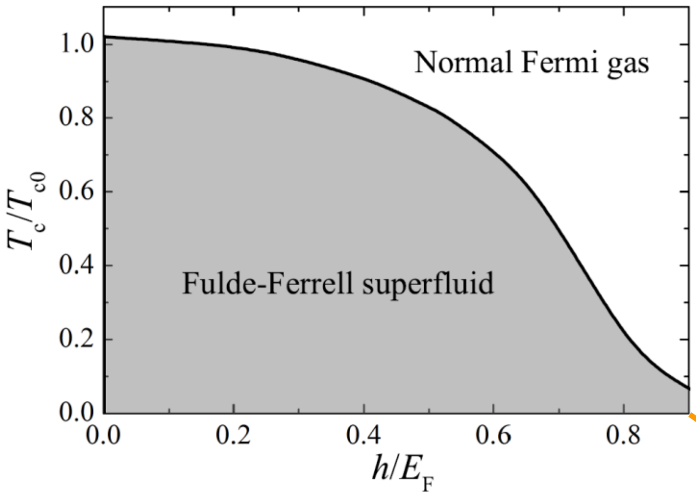
X.-J. Liu and HH, *Phys. Rev. A* **87**, 043616(R) (2013).

$$\Gamma\left(\mathbf{k},\omega\right)=\mathcal{A}_{\downarrow\downarrow}\left[\mathbf{k}+(k_R-\frac{q}{2})\mathbf{e}_z,\xi_{\mathbf{k}}-\omega\right]f\left(\xi_{\mathbf{k}}-\omega\right)$$



X.-J. Liu and HH, *Phys. Rev. A* **87**, 043616(R) (2013).

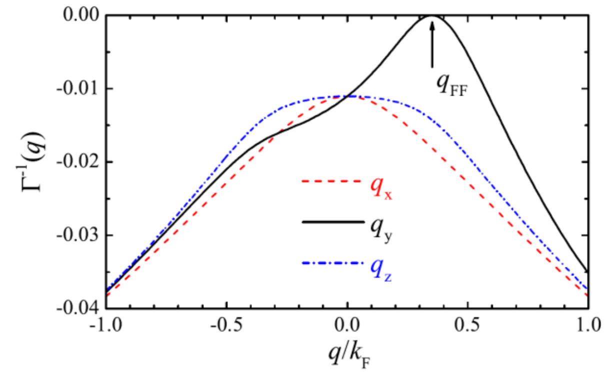
Fulde-Ferrell superfluid with Rashba SOC



finite-T mean-field phase diagram

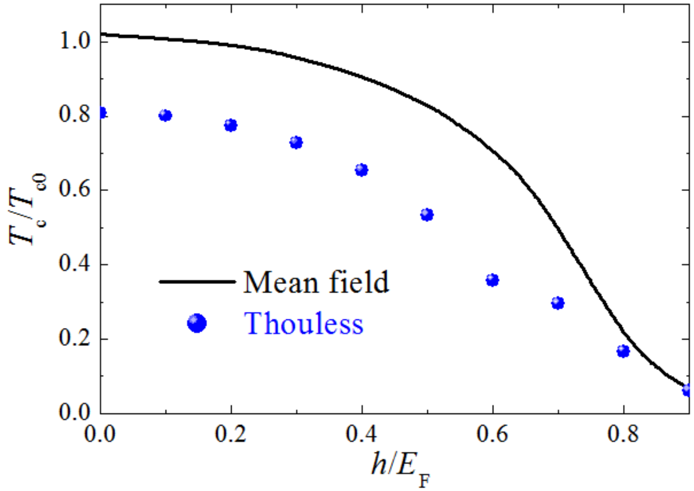
HH and X.-J. Liu, NJP **15**, 093037 (2013).

q_{max} **!= 0** indicates FF instability



$$\max \Gamma^{-1}(\mathbf{q}, i\nu_n = 0)\,|_{T=T_c} = 0$$

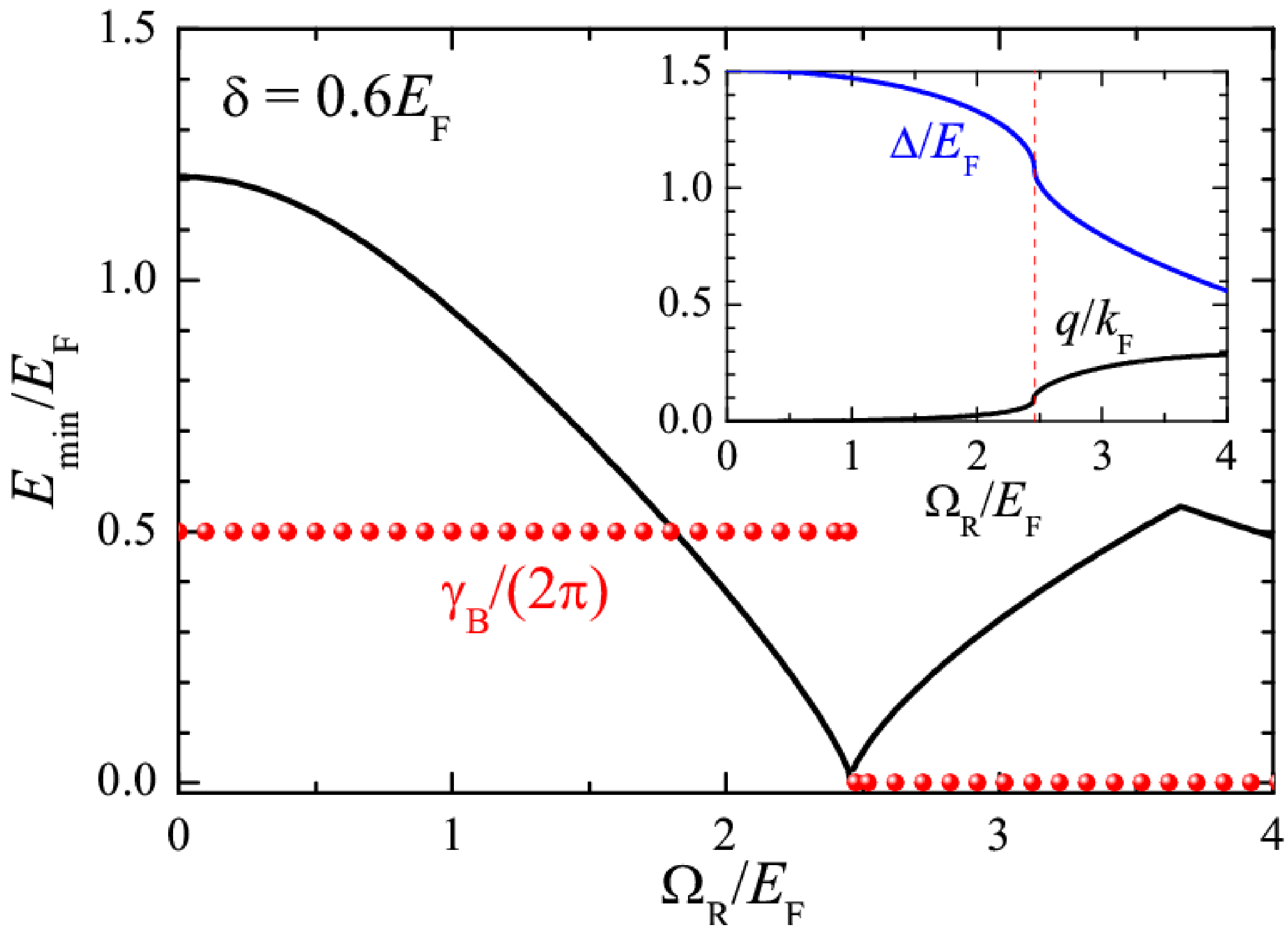
Thouless criterion leads to a better critical temperature.



X.-J. Liu, PRA **88**, 043607 (2013).

Topological Fulde-Ferrell superfluid (1D)

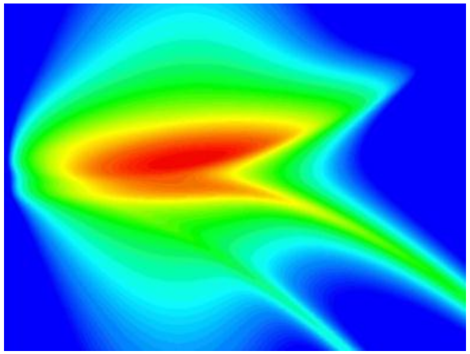
Topological + Fulde-Ferrell = Topological Fulde-Ferrell ?



X.-J. Liu and HH, *Phys. Rev. A* **88**, 023622 (2013).

Summary

Anisotropic superfluidity



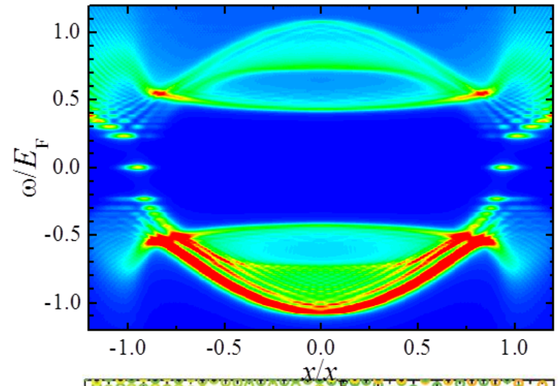
Our work:

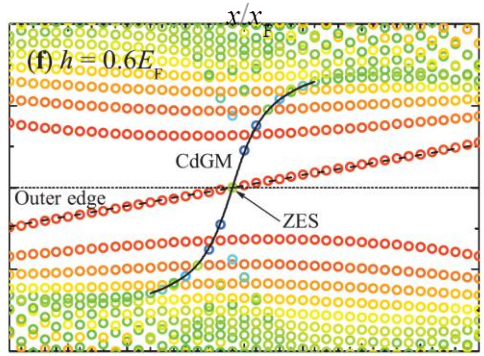
– PRL **107**, 195304 (2011);
– PRA **84**, 063618 (2011).

Others:

– Shenoy *et al*., PRB (2011);
– Iskin *et al*., PRL (2011);
– Sade Melo *et al*., PRA (2012);
– ……

(out-of-plane Zeeman field) Topological superfluidity





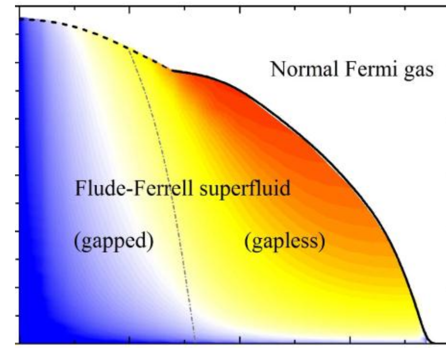
Our work:

– PRA **85**, 021603(R) (2012);
– PRA **85**, 033622 (2012);
– PRL **110**, 020401 (2013);
– PRA **87**, 013622 (2013).

Others:

– Mueller *et al*., PRA (2012);
– Sade Melo *et al*., PRL (2012);
– ……

(in-plane Zeeman field) Fulde-Ferrell superfluidity



Our work:

– PRA **87**, 043613(R) (2013);
– PRA **88**, 023622 (2013);
– PRA **88**, 043607 (2013);
– NJP **15**, 093037 (2013).

Others:

– C. Zhang *et al*., PRA (2013);
– Yi and Zhang *et al*., PRL (2013);
– Shenoy, PRA (2013);
– Pu *et al*., NJP (2013);
– Zhou *et al*., PRA (2013).

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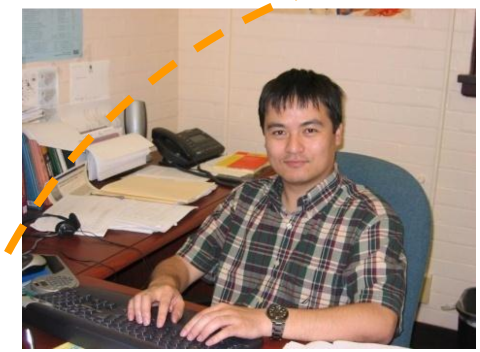
+



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Lin Dong

+

Jing Zhang *et al.*

Shanxi University

+

Shi-Guo Peng & Kaijun Jiang

Wuhan Institute of Physics…

Reference: review chapter

CHAPTER 2

FERMI GASES WITH SYNTHETIC SPIN–ORBIT COUPLING

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